

## A STRONG INTUITIONISTIC THEORY OF FUNCTIONALS

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**Abstract.** In this paper we construct a Beth model for intuitionistic functionals of high types and use it to create a relatively strong theory  $SLP$  containing intuitionistic principles for functionals, in particular, the theory of the "creating subject", axioms for lawless functionals and some versions of choice axioms.

We prove that the intuitionistic theory  $SLP$  is equiconsistent with a classical typed set theory  $TI$ , where the comprehension axiom for sets of type  $n$  is restricted to formulas with no parameters of types  $> n$ . We show that each fragment of  $SLP$  with types  $\leq s$  is equiconsistent with the corresponding fragment of  $TI$  and that it is stronger than the previous fragment of  $SLP$ . Thus, both  $SLP$  and  $TI$  are much stronger than the second order arithmetic. By constructing the intuitionistic theory  $SLP$  and interpreting in it the classical set theory  $TI$ , we contribute to the program of justifying classical mathematics from the intuitionistic point of view.

**§1. Introduction.** The program of justifying classical mathematics from the intuitionistic point of view was developed by several authors (see, for example, [11, 4, 1, 8]). One approach is to construct an axiomatic theory with intuitionistic principles where a classical set theory is interpretable. Myhill [11] showed the consistency of a classical analysis with respect to an intuitionistic analysis with Kripke schema. Bernini [1] constructed such an intuitionistic axiomatic theory with functionals of high types and interpreted a classical typed set theory in it. However, his theory turned out to be inconsistent [13]. The consistency of its modifications [2, 13] was not proven.

In this paper we construct a consistent modification of Bernini's theory [1] in the following steps using reverse mathematics approach. First we consider a basic axiomatic theory  $L$  with intuitionistic logic and construct a Beth model  $\mathcal{B}_s$  for its fragment  $L_s$  with types  $\leq s$ . The model  $\mathcal{B}_s$  is the generalisation to high-order functionals of the Beth model for choice sequences from [12].

Then we check what intuitionistic principles for high-order functionals hold in the Beth models  $\mathcal{B}_s (s \geq 1)$ . Combining those together we create a relatively strong intuitionistic theory  $SLP$ ; it includes axioms for lawless functionals, some versions of choice axioms and axioms for the "creating subject". This is reflected in the name:  $S$  is for strong,  $L$  is for lawless and  $P$  is for proofs, i.e. the theory

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of the "creating subject". The Beth model  $\mathcal{B}_s$  is used to prove the consistency of  $SLP$ .

Next we consider a sub-theory  $TI$  of the classical typed set theory  $TT$  and interpret  $TI$  in  $SLP$ .  $TI$  has a restricted comprehension axiom, but it is still quite strong; we believe that many parts of classical mathematics can be developed in  $TI$  and therefore justified from the intuitionistic point of view, due to the interpretation of  $TI$  in  $SLP$ .

In section 2 we remind the reader the general definition and basic properties of a Beth model. In section 3 we define the basic theory  $L$ . In section 4 we construct our Beth model for the fragment  $L_s (s \geq 1)$ . In section 5 we extend  $L$  to a theory  $LP$  by axioms for the "creating subject" and in section 6 we extend our Beth model to the fragment of  $LP$  with types  $\leq s$ . Section 7 is technical and contains lemmas about the Beth model used in further sections. In sections 8-11 we consider different intuitionistic principles in our Beth model. The principles that hold in the model are combined to create the theory  $SLP$  in section 12. In the rest of section 12 we interpret  $TI$  in  $SLP$ .

The original theory of Bernini [1] was intended to be a very strong intuitionistic theory. So in the last two sections we investigate the proof-theoretical strengths of our theories and show that  $SLP$  is relatively strong, though weaker than the full typed set theory  $TT$ . In section 13 we construct a truth predicate in the language  $TI$  and use it to prove that each fragment of  $TI$  with types  $\leq s$  is stronger than the previous fragment. Similarly, in section 14 we formalise the forcing relation of our Beth model in the language  $TI$  and show that  $SLP$  is equiconsistent with  $TI$  and that each fragment of  $SLP$  with types  $\leq s$  is stronger than the previous fragment. In particular,  $SLP$  is stronger than the second-order arithmetic.

In the rest of the Introduction we explain some notations and terminology.

In this paper we will use the same terminology as in the books by Kolmogorov and Dragalin [7, 3]; in particular, this relates to the following concepts of a logical-mathematical language and an axiomatic theory.

A *logical-mathematical language of first order* (or *logical-mathematical language*, or just *language* in short) is defined as  $\Omega = \langle Srt, Cnst, Fn, Pr \rangle$ , where

$Srt$  is a non-empty set of sorts of objects, and for each sort  $\pi \in Srt$  there is a countable collection of variables of this sort;

$Cnst$  is the set of all constants of the language;

$Fn$  is the set of all functional symbols of the language;

$Pr$  is the set of all predicate symbols of the language.

In the language  $\Omega$  we can construct terms, atomic formulas and formulas.

Terms of different sorts are constructed recursively from constants and variables using functional symbols. The complexity of a term  $t$  is the number of occurrences of functional symbols in  $t$ .

For each predicate symbol  $P(x_1, \dots, x_k)$  and terms  $t_1, \dots, t_k$  of corresponding sorts we construct an atomic formula  $P(t_1, \dots, t_k)$ .

Formulas are constructed from atomic formulas and logical constant  $\perp$  using logical connectives  $\wedge$  (conjunction),  $\vee$  (disjunction) and  $\supset$  (implication), and quantifiers  $\forall$  and  $\exists$ . The negation  $\neg\varphi$  is an abbreviation for  $\varphi \supset \perp$ . The logical connective  $\equiv$  (equivalence) is defined as  $(\varphi \supset \psi) \wedge (\psi \supset \varphi)$ . The complexity of

a formula  $\varphi$  is the number of occurrences of logical symbols (the main three connectives and quantifiers) in  $\varphi$ .

Next we define an *axiomatic theory* (or just *theory* in short) as  $Th = \langle \Omega, l, A \rangle$ , where each of the three objects is described as follows.

$\Omega$  is a logical-mathematical language.

$l$  is the logic of the theory. We will consider only two well-known logics: the classical logic CPC (classical predicate calculus) and the intuitionistic logic HPC (Heyting's predicate calculus); their detailed description can be found, for example, in [3].

$A$  is some set of closed formulas of the language  $\Omega$  called the set of non-logical axioms of  $Th$ . When axioms are stated as non-closed formulas, it means that they must be closed by universal quantifiers over all parameters.

The notation  $Th \vdash \varphi$  (formula  $\varphi$  is derivable in the theory  $Th$ ) means that  $\varphi$  is derivable in the logic  $l$  from a finite subset of the axiom set  $A$ . The theory  $Th$  is consistent if it is not true that  $Th \vdash \perp$ .

In this paper we consider axiomatic theories where variables have indices for types. We will often omit an index of a variable when its value is clear; usually it is the same index as in a previous occurrence of the variable.

## §2. Definitions.

**2.1. Definition of Beth model.** A *path* in a partially ordered set  $b$  is defined as a maximal linearly ordered subset of  $b$ . A *path through* an element  $x$  is defined as a path containing  $x$ .

The following algebraic definition of a Beth model is a slight modification of the definition by van Dalen [12].

**DEFINITION 2.1.** A **Beth model**  $\mathcal{B}$  for a logical-mathematical language  $\Omega$  is a quadruple of objects  $\langle M, \leq, D, Val \rangle$  described as follows.

$\langle M, \leq \rangle$  is a partially ordered set, called the *domain* of the model; elements of  $M$  are denoted  $\alpha, \beta, \gamma, \dots$ .

$D$  defines domains for variables of the language: for each sort  $\pi$ ,  $D_\pi$  is the domain for objects of the sort  $\pi$ .

We define an *evaluated formula* as a formula of the language  $\Omega$ , in which all parameters are replaced by objects from suitable domains. Another way to describe an evaluated formula is given in [12]: we assume that the language  $\Omega$  is extended by constants for all elements of  $D_\pi$  for each sort  $\pi$ ; then an evaluated formula is defined as a closed formula of the extended language.

$Val$  is a *valuation mapping* that assigns  $T$  or  $\perp$  to any couple  $(\alpha, \varphi)$ , where  $\alpha \in M$  and  $\varphi$  is an atomic evaluated formula, in such a way that this monotonicity condition holds:

$$(\beta \leq \alpha \& Val(\alpha, \varphi) = T) \Rightarrow Val(\beta, \varphi) = T.$$

Elements of  $M$  are interpreted as moments or worlds, the relation  $<$  on  $M$  is interpreted as "later in time".

A topological definition of a Beth model can be found in [3].

**DEFINITION 2.2.** In the Beth model a **forcing** relation  $\alpha \Vdash \varphi$  is defined for any  $\alpha \in M$  and evaluated formula  $\varphi$  by induction on the complexity of  $\varphi$ .

For an atomic formula  $\varphi$ ,  $\alpha \Vdash \varphi$  iff for any path  $S$  in  $M$  through  $\alpha$ , there is  $\beta \in S$  such that  $Val(\beta, \varphi) = T$ ;  
 $\alpha \nVdash \perp$  (falsity is never forced);  
 $\alpha \Vdash \psi \wedge \eta$  iff  $(\alpha \Vdash \psi \ \& \ \alpha \Vdash \eta)$ ;  
 $\alpha \Vdash \psi \vee \eta$  iff for any path  $S$  through  $\alpha$ , there is  $\beta \in S$  such that  $\beta \Vdash \psi$  or  $\beta \Vdash \eta$ ;  
 $\alpha \Vdash \psi \supset \eta$  iff for any  $\beta \leq \alpha$ ,  $(\beta \Vdash \psi \Rightarrow \beta \Vdash \eta)$ ;  
 $\alpha \Vdash \forall x \psi(x)$  iff for any element  $c$  of the appropriate domain,  $\alpha \Vdash \psi(c)$ ;  
 $\alpha \Vdash \exists x \psi(x)$  iff for any path  $S$  through  $\alpha$ , there is  $\beta \in S$  and an element  $c$  of the appropriate domain, such that  $\beta \Vdash \psi(c)$ .

We will assume that  $\langle M, \leq \rangle$  is a tree, which is the usual case.

For the Beth model  $\mathcal{B}$ ,  $\mathcal{B} \Vdash \varphi$  (an evaluated formula  $\varphi$  holds in  $\mathcal{B}$ , or forced in  $\mathcal{B}$ ) iff  $\varepsilon \Vdash \varphi$ , where  $\varepsilon$  is the root of the tree  $M$ .

**2.2. Facts about Beth models.** The following lemma and other facts about Beth models are explained in more detail in [12].

LEMMA 2.1. *A Beth model  $\mathcal{B}$  for a logical-mathematical language  $\Omega$  has the following properties.*

1. *Monotonicity:*  $(\alpha \Vdash \varphi \wedge \beta \leq \alpha) \Rightarrow \beta \Vdash \varphi$ .
2.  $\alpha \Vdash \varphi \Leftrightarrow$  *for any path  $S$  in  $M$  through  $\alpha$  there is  $\beta \in S$  with  $\beta \Vdash \varphi$ .*
3. *Soundness Theorem.* *For any closed formula  $\varphi$  of the language  $\Omega$ ,*  

$$(HPC \vdash \varphi) \Rightarrow (\mathcal{B} \Vdash \varphi).$$

We use classical logic in metamathematics and in this lemma, in particular.

Van Dalen [12] constructed a Beth model for the language of intuitionistic analysis *FIM*, which has two types of variables: over natural numbers and over choice sequences. We generalize this model to a Beth model for a language with many types of functionals.

**§3. Axiomatic theories  $L$  and  $L_s$ .** The language of theory  $L$  has the following variables:

- $x, y, z, \dots$  over natural numbers (variables of type 0),
- and variables of type  $n$  ( $n \geq 1$ ):  
 $F^n, G^n, H^n, \dots$  over  $n$ -functionals (functionals of type  $n$ );  
 $A^n, B^n, C^n, \dots$  over lawlike  $n$ -functionals;  
 $\mathcal{F}^n, \mathcal{G}^n, \mathcal{H}^n, \dots$  over lawless  $n$ -functionals.

Constants: 0 of type 0 and for each  $n \geq 1$  a constant  $K^n$  (an analog of 0 for type  $n$ ).

Functional symbols:  $N^n, Ap^n$  ( $n \geq 1$ ),  $S$  for successor,  $\cdot$  and  $+$ .

Predicate symbols:  $=_n$  for each  $n \geq 0$ .

Terms and  $n$ -functionals are defined recursively as follows.

1. Every numerical variable is a term.
2. Constant 0 is a term.

3. Every variable of type  $n$  is an  $n$ -functional.
4. Constant  $K^n$  is an  $n$ -functional.
5. If  $t$  is a term, then  $St$  is a term.
6. If  $t_1$  and  $t_2$  are terms, then  $t_1 + t_2$  and  $t_1 \cdot t_2$  are terms.
7. If  $Z$  is an  $n$ -functional, then  $N^n(Z)$  is an  $n$ -functional (a successor of  $Z$ ).
8. If  $Z$  is a 1-functional,  $t$  is a term, then  $Ap^1(Z, t)$  is a term.
9. If  $Z$  is an  $(n+1)$ -functional,  $t$  is a term, then  $Ap^{n+1}(Z, t)$  is an  $n$ -functional.

$Ap^n(Z, t)$  is interpreted as the result of application of functional  $Z$  to term  $t$ . We also denote  $Ap^n(Z, t)$  as  $Z(t)$ .

Here 1-functional is interpreted as a function from natural numbers to natural numbers, and  $(n+1)$ -functional is interpreted as a function from natural numbers to  $n$ -functionals.

As usual, lawlike (or constructive) functionals are interpreted as determined by laws or algorithms. Lawless functionals are opposite extreme: they could not be determined by any law. At each moment only an initial segment of a lawless functional is known, and there are no restrictions on its future values.

Atomic formulas:

- $t =_0 \tau$ , where  $t$  and  $\tau$  are terms,
- $Z =_n V$ , where  $Z$  and  $V$  are  $n$ -functionals ( $n \geq 1$ ).

Formulas are constructed from atomic formulas using logical connectives and quantifiers.

For a formula  $\varphi$ , its sort  $sort(\varphi)$  is the maximal type of parameters in  $\varphi$ ; it is 0 if  $\varphi$  has no parameters.

The theory  $L$  has intuitionistic predicate logic  $HPC$  with equality axioms and the following non-logical axioms.

1.  $Sx \neq 0$ ,  $Sx = Sy \supset x = y$ .
2.  $x + 0 = x$ ,  $x + Sy = S(x + y)$ .
3.  $x \cdot 0 = 0$ ,  $x \cdot Sy = x \cdot y + x$ .

The first four axioms define arithmetic at the bottom level. As usual, we introduce terms for all primitive recursive functions in the theory  $L$ .

4. Induction for natural numbers:  $\varphi(0) \wedge \forall x (\varphi(x) \supset \varphi(Sx)) \supset \forall x \varphi(x)$ , where  $\varphi$  is any formula of  $L$ .
5.  $K^{n+1}(x) = K^n$ ,  $\neg (N^n(F^n) = K^n)$ .

6.  $N^{n+1}(F^{n+1})(x) = N^n(F(x))$ ,  $N^n(F^n) = N^n(G^n) \supset F = G$ .

The axioms 5 and 6 describe  $K^n$  and  $N^n$  as analogs of zero and successor function, respectively, on level  $n$ .

7. Principle of primitive recursive completeness of lawlike functions:

$$\exists A^1 \forall x (A(x) = t),$$

where  $t$  is any term containing only variables of type 0 and variables over lawlike 1-functionals.

$L$  is a basic theory with  $n$ -functionals, which is a modification of Bernini's theory  $HL$  [1]. Later we extend it to a richer theory using reverse mathematics approach.

Denote  $L_s$  the fragment of the theory  $L$  with types not greater than  $s$ . The language of  $L_1$  has one type of functionals and is essentially the language of the intuitionistic analysis  $FIM$ .

**§4. Beth model for language  $L_s$ .** Any formal proof in the theory  $L$  is finite and therefore it is a proof in some fragment  $L_s$ . So instead of constructing a model for the entire language  $L$  with infinitely many types of functionals, we construct a Beth model  $\mathcal{B}_s$  for the fragment  $L_s$  ( $s \geq 1$ ) by generalizing the model of van Dalen [12].

**4.1. Notations.** A sequence of elements  $x_0, x_1, \dots, x_n$  is denoted  $x = \langle x_0, x_1, \dots, x_n \rangle$ , and we denote  $lth(x) = n + 1$ . Symbol  $*$  denotes the concatenation function:

$$\langle x_0, x_1, \dots, x_n \rangle * \langle y_0, y_1, \dots, y_k \rangle = \langle x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_k \rangle.$$

For a sequence  $x$ ,  $\langle x \rangle_n$  denotes its  $n^{th}$  element and  $\bar{x}(n)$  denotes the initial segment of  $x$  of length  $n$  for any  $n < lth(x)$ . So if  $x = \langle x_0, x_1, \dots, x_m \rangle$  and  $n \leq m$ , then  $\bar{x}(n) = \langle x_0, x_1, \dots, x_{n-1} \rangle$ .

For a function  $f$  on natural numbers,  $\bar{f}(n)$  denotes the initial segment of  $f$  of length  $n$ , that is the sequence  $\langle f(0), f(1), \dots, f(n-1) \rangle$ .

The notation  $f : a \dashrightarrow b$  means that  $f$  is a partial function from set  $a$  to set  $b$ .

For a function  $f$  of two variables denote  $f^{[x]} = \lambda y. f(x, y)$ .

The notation  $Z \downarrow$  means that the object  $Z$  is defined.

Next we introduce a few notations for a fixed set  $b$ .

- (a)  $b^{(n)}$  is the set of all sequences of elements of  $b$  that have length  $n$ .
- (b)  $b^*$  is the set of all finite sequences of elements of  $b$ .
- (c) On the set  $b^*$  a partial order  $\leq$  is defined by the following:

$$y \leq x \text{ iff } x \text{ is an initial segment of } y.$$

With this order  $b^*$  is a tree growing down; its root is the empty sequence  $\langle \rangle$ .

Suppose  $\langle d, \leq \rangle$  is a partially ordered set and  $f : (d \times \omega) \dashrightarrow c$ .

- (a)  $f$  is called *monotonic* on  $d$  iff for any  $x, y \in d$ ,  $(y \leq x \Rightarrow f^{[x]} \subseteq f^{[y]})$ .
- (b)  $f$  is called *complete* on  $d$  iff for any path  $S$  in  $d$ ,  $\bigcup \{f^{[x]} \mid x \in S\}$  is a total function on  $\omega$ .

**4.2. Definition of Beth model  $\mathcal{B}_s$ .** The following Beth model  $\mathcal{B}_s$  is a new, improved version of the Beth-Kripke model that we introduced in [6].

**DEFINITION 4.1.** We fix  $s \geq 1$  and define a Beth model  $\mathcal{B}_s$  for the language  $L_s$ :

$$\mathcal{B}_s = \langle M, \preceq, D, Val \rangle,$$

where each component is described as follows.

- 1) First we introduce a triple of objects  $\langle a_k, d_k, \preceq_k \rangle$  by induction on  $k$ ; here  $\preceq_k$  is a partial order on  $d_k$ .

$a_0 = \omega$ , the set of all natural numbers.

$d_0 = \{\langle x \rangle \mid x \in a_0^*\}$ ; thus, each element of  $d_0$  is a sequence of length 1.

$\preceq_0$  is generated by the order  $\leq$  on  $a_0^*$ :  $\langle x \rangle \preceq_0 \langle y \rangle$  iff  $x \leq y$ , that is the sequence  $y$  is an initial segment of the sequence  $x$ .

For  $\langle x \rangle \in d_0$  we denote  $lh(\langle x \rangle) = lth(x)$  and call it the *length* of  $\langle x \rangle$ .

For  $k \geq 1$ ,

$$a_k = \{f | f : (d_{k-1} \times \omega) \dashrightarrow a_{k-1}, f \text{ is monotonic and complete on } d_{k-1}\};$$

$$d_k = \bigcup_{m=0}^{\infty} \left( a_0^{(m)} \times a_1^{(m)} \times \dots \times a_k^{(m)} \right);$$

for  $x, y \in d_k$ ,  $x \preceq_k y$  iff for each  $i \leq k$ ,  $\langle x \rangle_i \leq \langle y \rangle_i$ , that is the sequence  $\langle y \rangle_i$  is an initial segment of the sequence  $\langle x \rangle_i$ .

If  $x \in a_0^{(m)} \times \dots \times a_k^{(m)}$ , we denote  $lh(x) = m$  and call it the *length* of  $x$ .

3) Domain of the Beth model is  $M = d_{s-1}$  with partial order  $\preceq_{s-1}$ , which we will denote just  $\preceq$ .

With this order  $M$  is a subtree of the direct product of trees  $a_0^*, \dots, a_{s-1}^*$ .  $M$  is a non-countable tree growing down. Its root is  $\varepsilon = \langle \underbrace{\langle \rangle, \dots, \langle \rangle}_s \rangle$ .

Elements of  $M$  are denoted  $\alpha, \beta, \gamma, \dots$ .

4) Domain for natural numbers is  $a_0 = \omega$ .

5) Domain for  $k$ -functionals is  $a_k$  ( $k = 1, 2, \dots, s$ ).

6) Domain for lawlike  $k$ -functionals:

$$b_k = \{f \in a_k | f^{[\langle \rangle]} \text{ is a total function}\}.$$

7) Before defining a domain for lawless functionals we introduce an auxiliary set  $c_k$  ( $k \geq 1$ ):

$$c_k = \{ \xi | \xi : (\omega \times a_{k-1}) \rightarrow a_{k-1} \text{ and } \forall n (\xi^{[n]} \text{ is a bijection on } a_{k-1}) \}.$$

Elements of  $c_k$  are called *k-permutations*. For any *k-permutation*  $\xi$  we define  $\nu_k(\xi)$  as the function  $f \in a_k$  such that for any  $n \in \omega$ ,  $x \in d_{k-1}$ :

$$f(x, n) = \begin{cases} \xi^{[n]}(\langle \langle x \rangle_{k-1} \rangle_n) & \text{if } n < lh(x), \\ \text{undefined} & \text{otherwise,} \end{cases}$$

Domain for lawless for  $k$ -functionals is  $l_k = \{\nu_k(\xi) | \xi \in c_k\}$ ,  $k = 1, 2, \dots, s$ .

8) It remains to define a valuation mapping  $Val$ . The definition consists of steps (a)-(d).

(a) For each constant  $Q$  of the language  $L$  we define its interpretation  $\widehat{Q}$ .

i)  $\widehat{0} = 0$ .

ii) The interpretation of constant  $K^k$  ( $k = 1, \dots, s$ ) is given by the following:

$$\widehat{K}^1(x, n) = 0 \text{ for any } n \in \omega \text{ and } x \in d_0;$$

$$\widehat{K}^{k+1}(x, n) = \widehat{K}^k \text{ for any } n \in \omega \text{ and } x \in d_k.$$

(b) For any  $\alpha \in M$  and functional symbol  $h$  of the language  $L_s$  we define its interpretation  $\widehat{h}^{[\alpha]}$  as follows.

- i) If  $\theta$  is  $\cdot$  or  $+$ , then  $\widehat{\theta}^{[\alpha]} = \theta$ .
- ii) Next we define a successor function  $S^k$  of type  $k$  ( $k = 0, 1, \dots, s$ ):  
 $S^0(x) = x + 1$ ;  $S^{n+1}(f) = S^n \circ f$ .  
 Functional symbol  $S$  is interpreted by:  $\widehat{S}^{[\alpha]} = S^0$ .  
 Functional symbol  $N^k$  is interpreted by:  $\widehat{N}^{k[\alpha]} = S^k$ .
- iii) Functional symbol  $Ap^k$  is interpreted by:  $\widehat{Ap}^{k[\alpha]}(f, n) = f(\bar{\alpha}(k), n)$  for any  $n \in \omega, f \in a_k, k = 1, \dots, s$ .  
 Thus, a  $k$ -functional  $f$  depends not on the entire node  $\alpha$  but only on its first  $k$  components.

We define an *evaluated term* as a term of the language  $L_s$ , in which all parameters are replaced by objects from suitable domains. We define an *evaluated  $k$ -functional* as a  $k$ -functional of the language  $L_s$ , in which all parameters are replaced by objects from suitable domains. These definitions are similar to the definition of an evaluated formula.

(c) Suppose  $Z$  is an evaluated term or  $n$ -functional. We define its interpretation  $Z^{[\alpha]}$  at node  $\alpha$  by induction on the complexity of  $Z$ .

- i) If  $Z$  is an evaluated variable, then  $Z^{[\alpha]} = Z$ .
- ii) If  $Z$  is a constant, then  $Z^{[\alpha]} = \widehat{Z}$ .
- iii) If  $Z = h(V_1, \dots, V_m)$ , where  $h$  is a functional symbol, then  
 $Z^{[\alpha]} = \widehat{h}^{[\alpha]}(V_1^{[\alpha]}, \dots, V_m^{[\alpha]})$ .

(d) The valuation mapping  $Val$  is defined by:

$$Val(\alpha, Z =_k V) = T \text{ iff } Z^{[\alpha]} \downarrow \& V^{[\alpha]} \downarrow \& Z^{[\alpha]} = V^{[\alpha]}, k = 0, 1, \dots, s.$$

This completes the definition of the model  $\mathcal{B}_s$ .

Clearly,  $b_k \subseteq a_k$  and  $l_k \subseteq a_k, k = 1, \dots, s$ . For  $\alpha \in M, lh(\alpha) = lth(\langle \alpha \rangle_k)$  for any  $k < s$ .

The main difference with the Beth model in [5] is in the definition of a  $k$ -functional and  $\widehat{Ap}^k$ : at each node  $\alpha$  the  $k$ -functional in the previous model depended only on one component  $\langle \alpha \rangle_{k-1}$ , while in the current model it depends on components  $\langle \alpha \rangle_0, \dots, \langle \alpha \rangle_{k-1}$ . This modification allows some versions of choice axioms to be forced in the new model, as will be shown further.

According to Definition 2.2, the forcing relation  $\alpha \Vdash \varphi$  is defined in the Beth model  $\mathcal{B}_s$ .

Next lemma states natural properties of the application predicate.

LEMMA 4.1. *Suppose  $k = 1, \dots, s, f \in a_k, n \in \omega$  and  $\widehat{Ap}^{k[\alpha]}(f, n) \downarrow$ . Then*

1.  $\widehat{Ap}^{k[\alpha]}(f, n) \in a_{k-1}$ ;
2.  $\beta \preceq \alpha \Rightarrow \widehat{Ap}^{k[\beta]}(f, n) \downarrow \& \widehat{Ap}^{k[\beta]}(f, n) = \widehat{Ap}^{k[\alpha]}(f, n)$ .



PROOF follows from the definition of  $\widehat{Ap}$  and monotonicity of the  $k$ -functional  $f$ .  $\dashv$

In the next lemma we show the validity of the interpretations of terms and functionals.

LEMMA 4.2. *Suppose  $\alpha \in M$ , and  $Z, V$  are evaluated  $k$ -functionals ( $k = 1, \dots, s$ ) or terms of the language  $L_s$  (in this case  $k = 0$ ). Then the following holds.*

1.  $(Z^{[\alpha]} \downarrow) \Rightarrow (Z^{[\alpha]} \in a_k)$ .
2.  $(Z^{[\alpha]} \downarrow) \& \beta \preceq \alpha \Rightarrow (Z^{[\beta]} \downarrow) \& (Z^{[\beta]} = Z^{[\alpha]})$ .
3.  $[Val(\alpha, Z =_k V) = T] \& \beta \preceq \alpha \Rightarrow [Val(\beta, Z =_k V) = T]$ .
4. *For any path  $S$  in  $M$  there is  $\gamma \in S$  such that  $Z^{[\gamma]} \downarrow$ .*

PROOF. Parts 1, 2 and 3 have the same proof as in [5]. Part 4 is proven by induction on the complexity of  $Z$ . We consider only the non-trivial case  $Z = Ap^{k+1}(V, t)$ . Consider a path  $S$  in  $M$ . By the inductive assumption there are  $\alpha, \beta \in S$  such that  $V^{[\alpha]} \downarrow$  and  $t^{[\beta]} \downarrow$ .

Denote  $f = V^{[\alpha]}, n = t^{[\beta]}$  and  $S' = \{\bar{\delta}(k+1) \mid \delta \in S\}$ . Then  $S'$  is a path in  $d_k$ ,  $f \in a_{k+1}$  and  $n \in \omega$  by part 1. Since the function  $f$  is complete, there is  $x \in S'$ , for which  $f^{[x]}(n) \downarrow$ .

For some  $\delta \in S$ ,  $x = \bar{\delta}(k+1)$ . Denote  $\gamma = \min\{\alpha, \beta, \delta\}$ . Then  $\gamma \in S$ .

By part 2,  $V^{[\gamma]} = V^{[\alpha]} = f, t^{[\gamma]} = t^{[\beta]} = n$  and  $\bar{\gamma}(k+1) \preceq_{k+1} \bar{\delta}(k+1) = x$ ,

since  $\gamma \preceq \delta$ . So  $Z^{[\gamma]} = V^{[\gamma]}(\bar{\gamma}(k+1), t^{[\gamma]}) = f(\bar{\gamma}(k+1), n) = f(x, n) = f^{[x]}(n)$ , since  $f$  is monotonic. Hence  $Z^{[\gamma]} \downarrow$ .  $\dashv$

Lemma 4.2.3 implies that  $\mathcal{B}_s$  is a Beth model. Clearly  $\mathcal{B}_s$  is embedded in  $\mathcal{B}_{s+1}$  and  $\mathcal{B}_1$  is essentially the van Dalen's model [12]. Our interpretation of a  $k$ -functional is similar to the interpretation of a choice sequence in [12]; the difference is in the recursive use of the tree  $d_{k-1}$  instead of the tree  $\omega^*$ .

LEMMA 4.3. *Suppose  $\alpha \in M, f \in l_k, n \in \omega$ . Then*

$$(f(n)^{[\alpha]} \downarrow) \Leftrightarrow (n < lh(\alpha)).$$

PROOF. It follows from the interpretation of a lawless functional.  $\dashv$

The following are some examples of  $k$ -functionals in the model  $\mathcal{B}_s, k = 1, \dots, s$ .

1.  $\widehat{K}^k$  is a lawlike  $k$ -functional; it is defined at the root of  $M$  and it is the same at every node;  $\widehat{K}^k \in b_k$ .

2. For  $x \in d_{k-1}, n \in \omega$ , define:

$$f(x, n) = \begin{cases} \langle \langle x \rangle_{k-1} \rangle_n & \text{if } n < lh(x), \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Clearly  $f = \nu_k(\xi)$ , where  $\xi$  is an element of  $c_k$  given by:  $\xi(n, y) = y$ . So  $f \in l_k$ ;  $f$  is a lawless  $k$ -functional whose values are determined by the path.

3. For  $x \in d_{k-1}, n \in \omega$ , define:

$$g(x, n) = \begin{cases} \text{undefined} & \text{if } lh(x) = 0, \\ \langle \langle x \rangle_{k-1} \rangle_0 & \text{if } lh(x) > 0. \end{cases}$$

The  $k$ -functional  $g$  is neither lawlike, nor lawless.

THEOREM 4.1 (Soundness for  $L_s$ ).

$$(L_s \vdash \varphi) \Rightarrow (\mathcal{B}_s \Vdash \bar{\varphi}),$$

where  $\bar{\varphi}$  is the closure of the formula  $\varphi$ , that is the formula  $\varphi$  with universal quantifiers over all its parameters.

PROOF was given in [5]. ⊣

**§5. Axiomatic theories  $LP$  and  $LP_s$ .** Since  $L$  contains arithmetic, in the language of  $L$  we can define Gödel numbering of symbols and expressions. For an expression  $q$  we will denote  $\ulcorner q \urcorner$  the Gödel number of  $q$  in this numbering. Detailed explanations of Gödel numbering can be found, for example, in [10].

The language of theory  $LP$  is the language of  $L$  with an extra predicate symbol  $P_{\ulcorner \bar{X} \urcorner, \varphi \urcorner}(z, \bar{X})$  for every formula  $\varphi$  of  $L$ , which has all its parameters in the list  $\bar{X}$ ; here  $\bar{X}$  denotes a list of variables  $X_1, \dots, X_n$ . According to tradition, we denote this symbol as  $\vdash_z \varphi(\bar{X})$ . It is interpreted that the formula  $\varphi(\bar{X})$  has been proven by the "creating subject" at time  $z$ .

Axioms of the theory  $LP$  are all axioms of  $L$ , where the axiom schemata are taken for the formulas of the new language, and the following axioms (CS1)-(CS3) for the "creating subject"; in all three of them  $\varphi$  is an arbitrary formula of  $L$ .

#### Axioms for the "creating subject".

$$(CS1) \ (\vdash_z \varphi) \vee \neg(\vdash_z \varphi).$$

$$(CS2) \ (\vdash_z \varphi) \supset (\vdash_{z+y} \varphi).$$

$$(CS3) \ \exists z(\vdash_z \varphi) \equiv \varphi.$$

The language of  $LP_s$  is the language of  $L_s$  with extra predicate symbols  $P_{\ulcorner \bar{X} \urcorner, \varphi \urcorner}(z, \bar{X})$ , where both  $\varphi$  and  $\bar{X}$  belong to the language of  $L_s$ . Axioms of the theory  $LP_s$  are all axioms of  $LP$ , which are formulas of the language  $LP_s$ .

If  $(A)$  is a formula of  $LP$ , we denote  $(A_s)$  the version of  $(A)$  with types not greater than  $s$ .

**§6. Extending the model  $\mathcal{B}_s$  to the language  $LP_s$ .** We extend the valuation mapping  $Val$  to each atomic formula  $\vdash_t \varphi$ , where  $t$  is an evaluated term and  $\varphi$  is an evaluated formula of  $L_s$ .

$$Val(\alpha, \vdash_t \varphi) = T \quad \text{iff} \quad t^{[\alpha]} \downarrow \& \exists \gamma \left[ (\alpha \preceq \gamma) \& (lh(\gamma) = t^{[\alpha]}) \& (\gamma \Vdash \varphi) \right].$$

LEMMA 6.1.

$$[Val(\alpha, \vdash_t \varphi) = T] \& (\beta \preceq \alpha) \Rightarrow [Val(\beta, \vdash_t \varphi) = T].$$

PROOF. Suppose  $Val(\alpha, \vdash_t \varphi) = T$  and  $\beta \preceq \alpha$ . Then there exists  $\gamma$  such that  $\alpha \preceq \gamma$ ,  $lh(\gamma) = t^{[\alpha]}$  and  $\gamma \Vdash \varphi$ . Since  $\beta \preceq \alpha \preceq \gamma$ , by Lemma 4.2.2,  $lh(\gamma) = t^{[\alpha]} = t^{[\beta]}$  and  $Val(\beta, \vdash_t \varphi) = T$ .  $\dashv$

The Lemma 6.1 implies that  $\mathcal{B}_s$  is a Beth model for the language  $LP_s$ .

LEMMA 6.2. *Suppose  $\alpha \in M$ ,  $Z$  is an evaluated term or functional, and  $\varphi$  is an evaluated formula of the language  $LP_s$ . Then*

$$(Z^{[\alpha]} \downarrow) \Rightarrow [\alpha \Vdash \varphi(Z) \Leftrightarrow \alpha \Vdash \varphi(Z^{[\alpha]})].$$

PROOF. It follows from statement 4.5.1 on pg. 85 of [3].  $\dashv$

THEOREM 6.1 (Soundness for  $LP_s$ ).

$$LP_s \vdash \varphi \Rightarrow \mathcal{B}_s \Vdash \bar{\varphi}.$$

PROOF. Proof is by induction on the length of derivation of  $\varphi$ .

For axioms and derivation rules of the intuitionistic logic HPC it follows from the Soundness Theorem for Beth models (Lemma 2.1.3). The axioms for equality hold in the model due to the definition of  $Val$ .

For the induction axiom (axiom 4) it easily follows from the definition of forcing. For axioms 1-3, 5-7 it follows from Theorem 4.1.

It only remains to check the three axioms for the "creating subject".

Proof for (CS1<sub>s</sub>)

Consider an arbitrary path  $S$  in  $M$ . There is an element  $\alpha \in S$  with  $lh(\alpha) = z$ . If  $\alpha \Vdash \varphi$ , then  $\alpha \Vdash (\vdash_z \varphi)$ .

Suppose  $\alpha \nVdash \varphi$ . Since  $lh(\alpha) = z$ , we have for any  $\beta \preceq \alpha$ ,  $\beta \nVdash (\vdash_z \varphi)$ . So  $\alpha \Vdash \neg(\vdash_z \varphi)$ .

This proves  $\mathcal{B}_s \Vdash (\vdash_z \varphi) \vee \neg(\vdash_z \varphi)$ .

Proof for (CS2<sub>s</sub>)

Suppose  $\alpha \Vdash (\vdash_z \varphi)$ . Consider an arbitrary path  $S$  in  $M$  through  $\alpha$  and the element  $\gamma \in S$  with  $lh(\gamma) = z$ . Then  $\gamma \Vdash \varphi$ .

Choose  $\beta \in S$  with  $lh(\beta) = z + y$ . Then  $\beta \preceq \gamma$  and by monotonicity of forcing  $\beta \Vdash \varphi$ . This implies  $\alpha \Vdash (\vdash_{z+y} \varphi)$ .

Proof for (CS3<sub>s</sub>)

$\Rightarrow$  Suppose  $\alpha \Vdash \exists z (\vdash_z \varphi)$ . Consider an arbitrary path  $S$  in  $M$  through  $\alpha$ . There are  $z \in \omega$  and  $\beta \in S$  such that  $\beta \Vdash (\vdash_z \varphi)$ . Then for some  $\delta \in S$  we have  $Val(\delta, \vdash_z \varphi) = T$ . So there is  $\gamma$  with  $\gamma \Vdash \varphi$  and  $\delta \preceq \gamma$ , hence  $\gamma \in S$ .

Thus, for any path  $S$  through  $\alpha$  there is  $\gamma \in S$  with  $\gamma \Vdash \varphi$ . By Lemma 2.1.2,  $\alpha \Vdash \varphi$ .

$\Leftarrow$  Suppose  $\alpha \Vdash \varphi$ . For any path  $S$  through  $\alpha$  we have:  $\alpha \in S$  and  $\alpha \Vdash (\vdash_z \varphi)$  for  $z = lh(\alpha)$ . Therefore  $\alpha \Vdash \exists z (\vdash_z \varphi)$ .  $\dashv$

Next we consider some intuitionistic principles in the language  $LP$  and check whether they hold in the model  $\mathcal{B}_s$ .

**§7. Permutations in the model  $\mathcal{B}_s$ .** This technical section contains some facts about the model  $\mathcal{B}_s$  necessary for the rest of the paper. We defined a  $k$ -permutation as an element of the set  $c_k$ , see Definition 4.1.

In this section we fix permutations  $\xi_0 \in c_1, \dots, \xi_{s-1} \in c_s$ . They generate a few mappings introduced in the following definition.

DEFINITION 7.1. Suppose  $k = 0, \dots, s-1$ .

1. Define  $\eta_k$  by the following:

$$\eta_k(x, n) = \left( \xi_k^{[n]} \right)^{-1}(x).$$

2. Define  $\tilde{\xi}_k$  by the following:

$$\tilde{\xi}_k(< x_0, \dots, x_m >) = < \xi_k^{[0]}(x_0), \dots, \xi_k^{[m]}(x_m) >.$$

3. Define  $\tilde{\xi}_k$  by the following:

$$\tilde{\xi}_k(< y_0, \dots, y_k >) = < \tilde{\xi}_0(y_0), \dots, \tilde{\xi}_k(y_k) >.$$

- 4.

$$\tilde{\xi} = \tilde{\xi}_{s-1}.$$

5. Define  $\Lambda_k$  by induction on  $k$  ( $k = 0, 1, \dots, s$ ).

$$\Lambda_0(n) = n.$$

Assume  $k \geq 1$  and  $\Lambda_{k-1}$  is defined. Define  $\Lambda_k(f) = g$ , where

$$g(x, n) = \Lambda_{k-1} \left( f \left( \tilde{\xi}_{k-1}(x), n \right) \right).$$

This completes the Definition 7.1.

Clearly,  $\tilde{\xi}_k$  is a bijection on  $a_k^*$  and  $\tilde{\eta}_k$  is its inverse. Also  $\tilde{\xi}_k$  is a bijection on  $d_k$  and  $\tilde{\eta}_k$  is its inverse. For a lawlike  $k$ -functional  $f$ ,  $\Lambda_k(f) = f$  because  $f(x, n)$  does not depend on  $x$ .

The following technical lemma gives reasons for introducing the aforementioned mappings.

LEMMA 7.1. Suppose  $k = 0, \dots, s-1$ .

1.  $\eta_k \in c_{k+1}$ .

2.  $\tilde{\xi}_k$  is an isomorphism of the partially ordered set  $d_k$ , i.e. a bijection preserving the order:

$$y \preceq_k x \Leftrightarrow \tilde{\xi}_k(y) \preceq_k \tilde{\xi}_k(x).$$

3.  $\tilde{\xi}$  is an isomorphism of the partially ordered set  $M$ , so

$$\beta \preceq \alpha \Leftrightarrow \tilde{\xi}(\beta) \preceq \tilde{\xi}(\alpha).$$

4. Suppose  $S$  is a path in  $d_k$  through  $x$  and  $S' = \{\tilde{\xi}_k(y) \mid y \in S\}$ . Then  $S'$  is a path in  $d_k$  through  $\tilde{\xi}_k(x)$ .

5. Suppose  $S$  is a path in  $M$  through  $\alpha$  and  $S' = \{\tilde{\xi}(\beta) \mid \beta \in S\}$ . Then  $S'$  is a path in  $M$  through  $\tilde{\xi}(\alpha)$ .

6. For  $k = 0, 1, \dots, s$ ,

$$f \in a_k \Leftrightarrow \Lambda_k(f) \in a_k.$$

PROOF. 1. Follows from the definition.

2. Clearly  $\tilde{\xi}_k$  is an isomorphism of the partially ordered set  $a_k^*$ , i.e. a bijection preserving the order:

$$y \leq x \Leftrightarrow \tilde{\xi}_k(y) \leq \tilde{\xi}_k(x).$$

The rest follows from the definition of  $\tilde{\xi}_k$ .

3. Follows from part 2 for  $k = s - 1$ .

4. Follows from part 2.

5. Follows from part 4 for  $k = s - 1$ .

6.  $\Rightarrow$  The proof is by induction on  $k$ . The basis step  $k = 0$  is obvious. We assume that the statement holds for  $k$  and prove it for  $k + 1$ .

Suppose  $f \in a_{k+1}$  and  $g = \Lambda_{k+1}(f)$ . To show that  $g \in a_{k+1}$  we check three conditions.

- (i) We need to prove that  $g : (d_k \times \omega) \dashrightarrow a_k$ .

Suppose  $x \in d_k$ ,  $n \in \omega$  and  $y = g(x, n)$  is defined. Then  $\tilde{\xi}_k(x) \in d_k$  and  $y = \Lambda_k(f(\tilde{\xi}_k(x), n))$  is defined. So  $f(\tilde{\xi}_k(x), n) \in a_k$  and by the inductive assumption  $y = \Lambda_k(f(\tilde{\xi}_k(x), n)) \in a_k$ .

- (ii) Monotonicity.

Suppose  $y \preceq_k x$  and  $g(x, n) \downarrow$ . Then  $f(\tilde{\xi}_k(x), n) \downarrow$  and  $\tilde{\xi}_k(y) \preceq_k \tilde{\xi}_k(x)$  by part 2. By monotonicity of  $f$ ,  $f(\tilde{\xi}_k(y), n) \downarrow$  and  $f(\tilde{\xi}_k(y), n) = f(\tilde{\xi}_k(x), n)$ , so  $g(y, n) \downarrow$  and  $g(y, n) = g(x, n)$ .

- (iii) Completeness.

Consider  $n \in \omega$  and a path  $S$  in  $d_k$ . Denote  $S' = \{\tilde{\xi}_k(z) \mid z \in S\}$ . By part 4,  $S'$  is a path in  $d_k$ . Then there is  $y \in S'$  such that  $f(y, n) \downarrow$ . For some  $z \in S$ ,  $y = \tilde{\xi}_k(z)$ . So  $f(\tilde{\xi}_k(z), n)$  is defined and belongs to  $a_k$ . By the inductive assumption  $g(x, n) = \Lambda_k(f(\tilde{\xi}_k(x), n)) \in a_k$  and  $g(x, n)$  is defined.

$\Leftarrow$  This follows from the previous if we use  $\eta_0, \dots, \eta_{s-1}$  instead of  $\xi_0, \dots, \xi_{s-1}$ .  $\dashv$

The following lemma describes a relation between permutations and forcing.

LEMMA 7.2. *Suppose:*

$\bar{X}$  is a list of variables  $X_1, \dots, X_r$  of types  $k_1, \dots, k_r$ , respectively;

$\bar{f}$  is a list  $f_1, \dots, f_r$  of objects from corresponding domains;

$\bar{g}$  is the list  $g_1, \dots, g_r$ , where  $g_i = \Lambda_{k_i}(f_i)$ ;

$\beta = \xi(\alpha)$ .

1. Suppose  $Z(\bar{X})$  is an  $n$ -functional ( $n = 1, \dots, s$ ) or a term (then  $n = 0$ ), and all its parameters are in the list  $\bar{X}$ . Then

$$i) (Z(\bar{f})^{[\beta]} \downarrow) \Rightarrow (Z(\bar{g})^{[\alpha]} \downarrow) \ \& \ [Z(\bar{g})^{[\alpha]} = \Lambda_n(Z(\bar{f})^{[\beta]})].$$

$$ii) (Z(\bar{f})^{[\beta]} \downarrow) \Leftrightarrow (Z(\bar{g})^{[\alpha]} \downarrow).$$

2. Suppose  $\varphi(\bar{X})$  is a formula of  $L_s$  and all its parameters are in the list  $\bar{X}$ . Then

$$\beta \Vdash \varphi(\bar{f}) \Leftrightarrow \alpha \Vdash \varphi(\bar{g}).$$

3. Suppose  $\varphi(\bar{X})$  is a formula of  $LP_s$  and all its parameters are in the list  $\bar{X}$ . Then

$$\beta \Vdash \varphi(\bar{f}) \Leftrightarrow \alpha \Vdash \varphi(\bar{g}).$$

PROOF. 1. i) It is proven by induction on the complexity of  $Z$ . We consider the case  $Z = Ap^{n+1}(V, t)$ . Other cases are straightforward. Suppose  $Z(\bar{f})^{[\beta]} \downarrow$ . Then

$$V(\bar{f})^{[\beta]} \downarrow \ \& \ t(\bar{f})^{[\beta]} \downarrow \ \& \ V(\bar{f})^{[\beta]} \left( \bar{\beta}(n), t(\bar{f})^{[\beta]} \right) \downarrow.$$

By the inductive assumption

$$V(\bar{g})^{[\alpha]} \downarrow \ \& \ t(\bar{g})^{[\alpha]} \downarrow \ \& \ V(\bar{g})^{[\alpha]} = \Lambda_{n+1} \left( V(\bar{f})^{[\beta]} \right) \ \& \ t(\bar{g})^{[\alpha]} = t(\bar{f})^{[\beta]}.$$

$$Z(\bar{g})^{[\alpha]} = V(\bar{g})^{[\alpha]} \left( \bar{\alpha}(n), t(\bar{g})^{[\alpha]} \right) = \Lambda_{n+1} \left( V(\bar{f})^{[\beta]} \right) \left( \bar{\alpha}(n), t(\bar{f})^{[\beta]} \right)$$

$$= \Lambda_n \left[ V(\bar{f})^{[\beta]} \left( \tilde{\xi}_n(\bar{\alpha}(n)), t(\bar{f})^{[\beta]} \right) \right] = \Lambda_n \left[ V(\bar{f})^{[\beta]} \left( \bar{\beta}(n), t(\bar{f})^{[\beta]} \right) \right]$$

$$= \Lambda_n \left( Z(\bar{f})^{[\beta]} \right).$$

So  $Z(\bar{g})^{[\alpha]} \downarrow$ .

ii) The implication from left to right follows from part i). The implication from right to left is proven using  $\eta_0, \dots, \eta_{s-1}$  instead of  $\xi_0, \dots, \xi_{s-1}$ .

2. The proof involves the same steps as in [5].

3. The proof is by induction on the complexity of  $\varphi$  using Definition 2.2 of forcing. Let us consider the case when  $\varphi$  is an atomic formula  $\vdash_{t(\bar{X})} \psi(\bar{X})$ . All other cases are proven the same way as in [5].

$\Rightarrow$  Suppose  $\beta \Vdash \varphi(\bar{f})$ . Consider an arbitrary path  $S$  in  $M$  through  $\alpha$ . Denote  $S' = \{\tilde{\xi}(\gamma) \mid \gamma \in S\}$ . By Lemma 7.1.5,  $S'$  is a path in  $M$  through  $\beta$ . So there is  $\delta' \in S'$  such that  $\text{Val}(\delta', \vdash_{t(\bar{f})} \psi(\bar{f})) = T$ . There exists  $\gamma'$  such that  $\delta' \preceq \gamma'$ ,  $lh(\gamma') = t(\bar{f})^{[\delta']}$  and  $\gamma' \Vdash \psi(\bar{f})$ . Clearly,  $\gamma' \in S'$  and for some  $\gamma \in S$ ,  $\gamma' = \tilde{\xi}(\gamma)$ . By part 2,  $\gamma \Vdash \psi(\bar{g})$ .

For some  $\delta \in S$ ,  $\delta' = \tilde{\xi}(\delta)$  and  $\delta \preceq \gamma$ . By the definition of  $\tilde{\xi}$  and part 1,  $lh(\gamma) = lh(\gamma') = t(\bar{f})^{[\delta']} = t(\bar{g})^{[\delta]}$ , so  $\text{Val}(\delta, \vdash_{t(\bar{g})} \psi(\bar{g})) = T$ . Hence  $\alpha \Vdash \varphi(\bar{g})$ .

$\Leftarrow$  follows from the previous if we take  $\eta_0, \dots, \eta_{s-1}$  instead of  $\xi_0, \dots, \xi_{s-1}$ .  $\dashv$

The following lemma states some properties of forcing in the model  $\mathcal{B}_s$ .

LEMMA 7.3. *Suppose:*

$\bar{X}$  is a list of variables  $X_1, X_2, \dots, X_r$ ;

$\bar{f}$  is a list  $f_1, f_2, \dots, f_r$  of objects from corresponding domains.

1. Suppose  $\varphi(\bar{X})$  is a formula of  $LP_s$  with all its parameters in the list  $\bar{X}$  and  $\text{sort}(\varphi) \leq m$ , where  $m \geq 1$ . Suppose

$$\bigwedge_{i=0}^{m-1} (\langle \alpha \rangle_i = \langle \beta \rangle_i).$$

Then

$$(\alpha \Vdash \varphi(\bar{f})) \Leftrightarrow (\beta \Vdash \varphi(\bar{f})).$$

2. Suppose  $\varphi(\mathcal{H}^p, \bar{X})$  is a formula of  $LP_s$  with all its parameters in the list  $\mathcal{H}^p, \bar{X}$ ;  $\text{sort}(\varphi) \leq p$  and  $\varphi$  does not have non-lawlike parameters of type  $p$  other than  $\mathcal{H}^p$ . Suppose  $g, h \in l_p$  and

$$(\forall n < lh(\alpha)) (g(n)^{[\alpha]} = h(n)^{[\alpha]}).$$

Then

$$(\alpha \Vdash \varphi(g, \bar{f})) \Leftrightarrow (\alpha \Vdash \varphi(h, \bar{f})).$$

PROOF. Denote  $k_i$  the type of variable  $X_i$ ,  $i = 1 \dots, r$ .

1. Denote  $l = lh(\alpha)$ . Since  $\langle \alpha \rangle_0 = \langle \beta \rangle_0$ , we have  $lh(\beta) = lh(\alpha) = l$ . For any  $k = 0, \dots, s-1$  define  $\xi_k$  by the following: for  $n \in \omega, x \in a_k$ ,

$$\xi_k(n, x) = \begin{cases} \langle \langle \beta \rangle_k \rangle_n & \text{if } (n < l \& x = \langle \langle \alpha \rangle_k \rangle_n), \\ \langle \langle \alpha \rangle_k \rangle_n & \text{if } (n < l \& x = \langle \langle \beta \rangle_k \rangle_n), \\ x & \text{otherwise.} \end{cases}$$

Then each  $\xi_k \in c_{k+1}$  and  $\beta = \tilde{\xi}(\alpha)$ . Denote  $g_i = \Lambda_{k_i}(f_i)$  for  $i = 1, \dots, r$ . By Lemma 7.2.3,

$$(1) \quad \beta \Vdash \varphi(\bar{f}) \Leftrightarrow \alpha \Vdash \varphi(\bar{g}).$$

If  $k < m$ , then  $\langle \alpha \rangle_k = \langle \beta \rangle_k$  and

$$\begin{aligned} (\forall n \in \omega)(\forall x \in a_k) \left( \xi_k^{[n]}(x) = x \right), \\ (\forall y \in d_k) \left( \tilde{\xi}_k(y) = y \right), \end{aligned}$$

so for any  $j \leq m$ ,

$$(\forall z \in a_j) (\Lambda_j(z) = z).$$

Since  $\text{sort}(\varphi) \leq m$ , for each  $i = 1, \dots, r$ , we have  $k_i \leq m$  and  $g_i = \Lambda_{k_i}(f_i) = f_i$ . So by (1)

$$\beta \Vdash \varphi(\bar{f}) \Leftrightarrow \alpha \Vdash \varphi(\bar{f}).$$

2. For some  $\chi_1, \chi_2 \in c_p$ ,  $g = \nu_p(\chi_1)$  and  $h = \nu_p(\chi_2)$ . Clearly  $p \geq 1$ . Define permutations  $\xi_0, \dots, \xi_{s-1}$ :

$$\xi_{p-1}^{[n]} = \left( \chi_2^{[n]} \right)^{-1} \circ \chi_1^{[n]}; \quad \xi_k^{[n]}(x) = x \text{ for } k \neq p-1.$$

By Lemma 7.2.3, it is sufficient to prove the following:

$$(2) \quad \tilde{\xi}(\alpha) = \alpha;$$

$$(3) \quad \Lambda_{k_i}(f_i) = f_i, \quad i = 1, \dots, r;$$

$$(4) \quad \Lambda_p(h) = g.$$

Proof of (2).

Denote  $\alpha = \langle \alpha_0, \dots, \alpha_{s-1} \rangle$ . Then  $\tilde{\xi}(\alpha) = \langle \tilde{\xi}_0(\alpha_0), \dots, \tilde{\xi}_{s-1}(\alpha_{s-1}) \rangle$ . Clearly  $\tilde{\xi}_k(\alpha_k) = \alpha_k$  for  $k \neq p-1$ .

Now consider  $k = p-1$ . Fix  $n < lh(\alpha)$ . Since  $g = \nu_p(\chi_1)$  and  $h = \nu_p(\chi_2)$ ,

$$\chi_1^{[n]}(\langle \alpha_{p-1} \rangle_n) = g(\bar{\alpha}(p), n) = g(n)^{[\alpha]} = h(n)^{[\alpha]} = h(\bar{\alpha}(p), n) = \chi_2^{[n]}(\langle \alpha_{p-1} \rangle_n).$$

Thus,  $\chi_1^{[n]}(\langle \alpha_{p-1} \rangle_n) = \chi_2^{[n]}(\langle \alpha_{p-1} \rangle_n)$  and

$$\xi_{p-1}^{[n]}(\langle \alpha_{p-1} \rangle_n) = \left( \chi_2^{[n]} \right)^{-1} \left( \chi_1^{[n]}(\langle \alpha_{p-1} \rangle_n) \right) = \langle \alpha_{p-1} \rangle_n.$$

So for any  $n < lh(\alpha)$ ,  $\xi_{p-1}^{[n]}(\langle \alpha_{p-1} \rangle_n) = \langle \alpha_{p-1} \rangle_n$  and  $\tilde{\xi}_{p-1}(\alpha_{p-1}) = \alpha_{p-1}$ .

Proof of (3).

For  $k < p-1$ ,  $\tilde{\xi}_k(x) = x$  for any  $x \in d_k$  and

$$(5) \quad (\forall y \in a_j) (\Lambda_j(y) = y), \quad j < p.$$

If  $f_i$  is a domain object for a lawlike  $k_i$ -functional, then  $\Lambda_{k_i}(f_i) = f_i$ ; otherwise  $k_i < p$  and  $\Lambda_{k_i}(f_i) = f_i$  by (5).

Proof of (4).



For any  $n \in \omega, x \in d_{p-1}$ ,

$$\Lambda_p(h)(x, n) = \Lambda_{p-1} \left( h \left( \tilde{\xi}_{p-1}(x), n \right) \right) = h \left( \tilde{\xi}_{p-1}(x), n \right) \text{ by (5).}$$

Since  $\tilde{\xi}_{p-1}(x)$  has the same length as  $x$ , it is sufficient to prove:

$$h \left( \tilde{\xi}_{p-1}(x), n \right) = g(x, n) \text{ for any } n < lh(x).$$

Since  $g = \nu_p(\chi_1)$  and  $h = \nu_p(\chi_2)$ , we have:

$$\begin{aligned} h \left( \tilde{\xi}_{p-1}(x), n \right) &= \chi_2^{[n]} \left( \langle \langle \tilde{\xi}_{p-1}(x) \rangle_{p-1} \rangle_n \right) = \chi_2^{[n]} \left( \xi_{p-1}^{[n]} (\langle \langle x \rangle_{p-1} \rangle_n) \right) \\ &= \chi_2^{[n]} \left[ \left( \chi_2^{[n]} \right)^{-1} \circ \chi_1^{[n]} (\langle \langle x \rangle_{p-1} \rangle_n) \right] = \chi_1^{[n]} (\langle \langle x \rangle_{p-1} \rangle_n) = g(x, n). \end{aligned}$$

□

**§8. Axioms for lawless functionals.** We can introduce finite sequences in  $LP$  and for any  $k$ -functional  $F$  use the notation  $\bar{F}(n) = \langle F(0), \dots, F(n-1) \rangle$ .

The following are modifications for the language  $LP$  of the axioms for lawless sequences in [12] and the axioms for lawless functionals in [1].

(LL1) The axiom of existence of lawless functionals:

$$\exists \mathcal{F}^n (\forall y \leq x) (\mathcal{F}(y) = F^n(y)).$$

(LL2)  $\mathcal{F}^n = \mathcal{G}^n \vee \mathcal{F}^n \neq \mathcal{G}^n$ .

(LL3) The principle of open data:

$$\varphi(\mathcal{H}^n) \supset \exists x \forall \mathcal{G}^n [\bar{\mathcal{G}}(x) = \bar{\mathcal{H}}(x) \supset \varphi(\mathcal{G})],$$

where  $\varphi$  is a formula of  $LP$ ,  $sort(\varphi) \leq n$  and  $\varphi$  does not have non-lawlike parameters of type  $n$  other than  $\mathcal{H}^n$ .

Denote (LL) a conjunction of the closures of (LL1), (LL2) and (LL3).

THEOREM 8.1.  $\mathcal{B}_s \Vdash (LL_s)$ .

PROOF. (LL1<sub>s</sub>). Fix  $f \in a_n$  and a path  $S$  in  $M$ . By induction on  $x$  we will prove:

$$(\exists \alpha \in S) (\forall y \leq x) f(y)^{[\alpha]} \downarrow.$$

For  $x = 0$  it follows from Lemma 4.2.4.

Assume it holds for  $x$ : for some  $\beta \in S$ ,  $(\forall y \leq x) f(y)^{[\beta]} \downarrow$ . By Lemma 4.2.4, there is  $\gamma \in S$  such that  $f(x+1)^{[\gamma]} \downarrow$ . For  $\alpha = \min\{\beta, \gamma\}$  we have  $(\forall y \leq x+1) f(y)^{[\alpha]} \downarrow$ .

Now let us prove that (LL1<sub>s</sub>) holds in the model. Fix an arbitrary  $x \in \omega$  and  $\alpha \in S$  such that  $(\forall y \leq x) f(y)^{[\alpha]} \downarrow$ .

The path  $S = \{\langle \bar{g}_0(k), \dots, \bar{g}_{s-1}(k) \rangle \mid k \in \omega\}$  for some functions

$g_0 : \omega \rightarrow a_0, \dots, g_{s-1} : \omega \rightarrow a_{s-1}$ . Denote  $\beta = \langle \bar{g}_0(x+1), \dots, \bar{g}_{s-1}(x+1) \rangle$

and  $\gamma = \min\{\alpha, \beta\}$ . Then  $\beta, \gamma \in S$  and  $(\forall y \leq x) f(y)^{[\gamma]} \downarrow$ .

Define  $\xi$  by:

$$\xi(y, z) = \begin{cases} f(y)^{[\gamma]} & \text{if } y \leq x \& z = g_{n-1}(y), \\ g_{n-1}(y) & \text{if } y \leq x \& z = f(y)^{[\gamma]}, \\ z & \text{otherwise.} \end{cases}$$

For any  $y \leq x$ ,  $g_{n-1}(y) \in a_{n-1}$  and  $f(y)^{[\gamma]} \in a_{n-1}$ . So  $\xi : (\omega \times a_{n-1}) \rightarrow a_{n-1}$  and for any  $y \in \omega$ ,  $\xi^{[y]}$  is a bijection on  $a_{n-1}$ . Therefore  $\xi \in c_n$ . Take  $h = \nu_n(\xi) \in l_n$ . It remains to prove:

$$\gamma \Vdash (\forall y \leq x) [h(y) = f(y)].$$

For any  $y \leq x$ ,  $h(y)^{[\gamma]} = h(\bar{\gamma}(n), y) = \xi^{[y]}(\langle \langle \gamma \rangle_{n-1} \rangle_y)$ . Since  $\gamma \preceq \beta$ , we have  $\langle \gamma \rangle_{n-1} \leq \langle \beta \rangle_{n-1}$  and  $\langle \langle \gamma \rangle_{n-1} \rangle_y = g_{n-1}(y)$ . So  $h(y)^{[\gamma]} = \xi^{[y]}(g_{n-1}(y)) = f(y)^{[\gamma]}$  by the definition of  $\xi$ .

For  $(LL2_s)$  the proof follows from the interpretation of equality.

$(LL3_s)$ . Suppose  $\bar{X}$  is a list of variables  $X_1, \dots, X_r$  and  $\bar{f}$  is a list  $f_1, \dots, f_r$  of objects from corresponding domains. Suppose all parameters of  $\varphi$  are in the list  $\mathcal{H}^n, \bar{X}$ . Suppose  $h \in l_n$  and

$$(6) \quad \alpha \Vdash \varphi(h, \bar{f}).$$

Take  $x = lh(\alpha)$ . It is sufficient to prove that for any  $g \in l_n$ ,

$$\alpha \Vdash (\forall y < x) (g(y) = h(y)) \supset \varphi(g, \bar{f}).$$

Suppose  $\beta \preceq \alpha$  and  $(\forall y < x) \beta \Vdash (g(y) = h(y))$ . Since  $x = lh(\alpha)$ , we have  $(\forall y < x) [g(y)^{[\alpha]} \downarrow \& h(y)^{[\alpha]} \downarrow]$  by Lemma 4.3. So  $(\forall y < x) (g(y)^{[\alpha]} = h(y)^{[\alpha]})$ .

By (6) and Lemma 7.3.2,  $\alpha \Vdash \varphi(g, \bar{f})$ .  $\dashv$

**§9. Choice axioms in the model  $\mathcal{B}_s$ .** For an  $m$ -functional  $F$  of language  $LP$  and  $n \leq m$  denote  $F \underbrace{(0) \dots (0)}_n$  as  $F(0)^n$  for brevity.

We consider the following two axioms of choice and show that they hold in the model  $\mathcal{B}_s$ .

(C1) The axiom of choice for numbers:

$$\forall x \exists y \varphi(x, y) \supset \exists F^m \forall x \varphi(x, F(x)(0)^{m-1}),$$

where  $\varphi$  is a formula of  $LP$  and  $m \geq \max(\text{sort}(\varphi), 1)$ .

(C2) The axiom of choice with uniqueness:

$$\forall x \exists! G^n \varphi(x, G) \supset \exists F^m \forall x \varphi(x, F(x)(0)^{m-n-1}),$$

where  $\varphi$  is a formula of  $LP$  and  $m \geq \max(\text{sort}(\varphi), n + 1)$ .

Denote  $(C)$  a conjunction of the closures of (C1) and (C2).

**THEOREM 9.1.**  $\mathcal{B}_s \Vdash (C_s)$ .

PROOF. In case  $s = 1$  the proof is the same as in [12], since the model described there is essentially  $\mathcal{B}_1$ . We will assume  $s \geq 2$ .

For  $a \in a_0^{(k)} \times \dots \times a_{m-1}^{(k)}$ , denote:

$$\text{extension}(a) = a * \langle \underbrace{\hat{K}^m, \dots, \hat{K}^m}_k, \dots, \underbrace{\hat{K}^{s-1}, \dots, \hat{K}^{s-1}}_k \rangle \rangle.$$

The operation of extension extends a node  $a \in d_{m-1}$  by constant functionals to a node in the domain  $M$ ; it preserves the length of the node.

By Lemma 7.3.1, for any formula  $\chi$  with  $\text{sort}(\chi) \leq m$  and its evaluation  $\tilde{\chi}$ :

$$(7) \quad \bar{\alpha}(m) = \bar{\beta}(m) \Rightarrow (\alpha \Vdash \tilde{\chi} \Leftrightarrow \beta \Vdash \tilde{\chi}).$$

PROOF for  $(C1_s)$

Denote  $\psi(x, y)$  the formula  $\varphi(x, y)$ , in which all parameters, except  $x$  and  $y$ , are replaced by objects from corresponding domains.

Denote

$$\text{shortest}(\alpha, \beta, x) \Leftarrow \alpha \preceq \beta \& \exists y [\beta \Vdash \psi(x, y)] \& (\forall \gamma \succ \beta) \neg \exists z [\gamma \Vdash \psi(x, z)].$$

We have:

$$(8) \quad \exists y [\alpha \Vdash \psi(x, y)] \Rightarrow \exists! \beta \text{shortest}(\alpha, \beta, x).$$

Indeed, of the finite number of nodes  $\beta \succ \alpha$  with  $\exists y [\beta \Vdash \psi(x, y)]$  we take the unique  $\beta$  with smallest length.

Fix  $\alpha \in M$ . Suppose

$$\alpha \Vdash \forall x \exists y \psi(x, y).$$

Denote  $\alpha' = \text{extension}(\bar{\alpha}(m))$ . So by (7),

$$(9) \quad \alpha' \Vdash \forall x \exists y \psi(x, y)$$

CASE  $m = 2$ . First we construct the proof for this case because it clearly demonstrates the main construction. Other cases will be considered later.

Thus,  $\text{sort}(\varphi) \leq 2$ .

By (7), it is sufficient to prove:

$$(10) \quad \alpha' \Vdash \exists F^2 \forall x \psi(x, F(x)(0)).$$

First we construct a function  $f \in a_2$  that can be used in (10).

For fixed  $\beta$  and  $x \in \omega$  define function  $q_{\beta, x}$ :

$$q_{\beta, x}(v, k) = \begin{cases} 0 & \text{if } lh(v) \geq lh(\beta) \& \neg(v \preceq_0 \bar{\beta}(1)), \\ \text{the smallest number } y, \text{ for which } \beta \Vdash \psi(x, y) & \\ & \text{if } v \preceq_0 \bar{\beta}(1) \& \exists z [\beta \Vdash \psi(x, z)], \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Next we construct a function that can be used in (10).

$$f(u, x) = \begin{cases} \widehat{K}^1 & \text{if } lh(u) \geq lh(\alpha) \& \neg(u \preceq_1 \bar{\alpha}(2)), \\ q_{\beta, x} & \text{for the unique } \beta \text{ with } shortest(extension(u), \beta, x) \\ & \text{if } u \preceq_1 \bar{\alpha}(2) \& \exists y [extension(u) \Vdash \psi(x, y)], \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Proof that  $f \in a_2$ .

1) First we prove that  $f : (d_1 \times \omega) \dashrightarrow a_1$ . Suppose  $f(u, x) \downarrow$ . We need to prove that  $f(u, x) \in a_1$ .

When  $f(u, x) = \widehat{K}^1$ , it is obvious. Assume  $f(u, x) = q_{\beta, x}$ . Then  $u \preceq_1 \bar{\alpha}(2)$ ,  $shortest(extension(u), \beta, x)$  and

$$(11) \quad \exists z [\beta \Vdash \psi(x, z)].$$

The proof that  $q_{\beta, x} \in a_1$  consists of three parts.

a) Clearly,  $q_{\beta, x} : (d_0 \times \omega) \dashrightarrow a_0$ .

b) Monotonicity of  $q_{\beta, x}$ .

Suppose  $q_{\beta, x}(v, k) \downarrow$  and  $w \preceq_0 v$ . Then  $lh(v) \geq lh(\beta)$ . There are two cases.

i)  $\neg(v \preceq_0 \bar{\beta}(1))$ .

If  $w \preceq_0 \bar{\beta}(1)$ , then  $\bar{\beta}(1) = \bar{w}(m)$  and  $v = \bar{w}(n)$  for some  $m$  and  $n$ , which implies  $v \preceq_0 \bar{\beta}(1)$  due to  $lh(v) \geq lh(\beta)$ . Thus,  $\neg(w \preceq_0 \bar{\beta}(1))$  and  $q_{\beta, x}(w, k) = 0 = q_{\beta, x}(v, k)$ .

ii)  $v \preceq_0 \bar{\beta}(1)$ .

Then  $w \preceq_0 \bar{\beta}(1)$ . By (11) both  $q_{\beta, x}(w, k)$  and  $q_{\beta, x}(v, k)$  equal the smallest number  $y$ , for which  $\beta \Vdash \psi(x, y)$ .

c) Completeness of  $q_{\beta, x}$ .

Consider a path  $S_0$  in  $d_0$  and  $k \in \omega$ . Take  $v \in S_0$  with length  $lh(\beta)$ . There are two cases.

i)  $\bar{\beta}(1) \notin S_0$ .

Then  $\neg(v \preceq_0 \bar{\beta}(1))$  and  $q_{\beta, x}(v, k) = 0$ .

ii)  $\bar{\beta}(1) \in S_0$ .

Then  $v = \bar{\beta}(1)$  and by (11),  $q_{\beta, x}(v, k)$  equals the smallest number  $y$ , for which  $\beta \Vdash \psi(x, y)$ .

This completes the proof that  $q_{\beta, x} \in a_1$  and therefore  $f : (d_1 \times \omega) \dashrightarrow a_1$ .

2) Monotonicity of  $f$ .

Suppose  $f(u, x) \downarrow$  and  $v \preceq_1 u$ . Then  $lh(v) \geq lh(u) \geq lh(\alpha)$ . There are two cases.

i)  $\neg(u \preceq_1 \bar{\alpha}(2))$ .

In this case  $\neg(v \preceq_1 \bar{\alpha}(2))$ , so  $f(v, x) = \widehat{K}^1 = f(u, x)$ .

ii)  $u \preceq_1 \bar{\alpha}(2)$ .

Then  $v \preceq_1 \bar{\alpha}(2)$ , there is a unique  $\beta$  with  $shortest(extension(u), \beta, x)$  and  $f(u, x) = q_{\beta, x}$ .

For some  $y$  we have:  $extension(u) \Vdash \psi(x, y)$ .

Since  $extension(v) \preceq extension(u)$ , we have:  $shortest(extension(v), \beta, x)$  and by monotonicity of forcing,  $extension(v) \Vdash \psi(x, y)$ . So  $f(v, x) = q_{\beta, x}$  for the same  $\beta$ .

3) Completeness of  $f$ .

Consider  $x \in \omega$  and a path  $S_1$  in  $d_1$ . There are two cases.

i)  $\bar{\alpha}(2) \notin S_1$ .

Take  $u \in S_1$  with length  $lh(\alpha)$ . Then  $\neg(u \preceq_1 \bar{\alpha}(2))$  and  $f(u, x) = \widehat{K}^1$ .

ii)  $\bar{\alpha}(2) \in S_1$ .

Denote  $S'_1 = \{extension(v) \mid v \in S_1\}$ . Then  $S'_1$  is a path in  $M$  through  $\alpha'$  and by (9), for some  $\delta \in S'_1$ ,  $\exists y [\delta \Vdash \psi(x, y)]$ .

Take  $\gamma = \min\{\alpha', \delta\}$  and  $u = \bar{\gamma}(2)$ . Then  $u \preceq_1 \bar{\alpha}(2)$ ,  $\gamma = extension(u)$ ,  $u \in S_1$  and  $\exists y [\gamma \Vdash \psi(x, y)]$ . By (8), there is a unique  $\beta$  with  $shortest(\gamma, \beta, x)$ . Therefore  $f(u, x) = q_{\beta, x}$ . This completes the proof of  $f \in a_2$ .

Proof of (10).

It is sufficient to prove:

$$\forall x [\alpha' \Vdash \psi(x, f(x)(0))].$$

Fix  $x \in \omega$ . By Lemma 2.1.2, it is sufficient to prove:

$$(\forall \text{ path } S_2 \text{ through } \alpha') (\exists \gamma \in S_2) [\gamma \Vdash \psi(x, f(x)(0))].$$

Consider a path  $S_2$  in  $M$  through  $\alpha'$ . Denote  $S'_2 = \{extension(\bar{\sigma}(2)) \mid \sigma \in S_2\}$ . Then  $S'_2$  is also a path in  $M$  through  $\alpha'$ .

By (9), for some  $\delta' \in S'_2$ ,  $\exists y [\delta' \Vdash \psi(x, y)]$ . Denote  $\delta = \min\{\alpha', \delta'\}$  and  $u = \bar{\delta}(2)$ . Then  $\delta \in S'_2$ ,  $u \preceq_1 \bar{\alpha}(2)$ ,  $\delta = extension(u)$  and  $\exists y [\delta \Vdash \psi(x, y)]$ .

By (8), for some  $\beta$  we have:  $shortest(\delta, \beta, x)$ ,  $\delta \preceq \beta$  and  $f(x)^{[\delta]} = f(u, x) = q_{\beta, x}$ .

Therefore  $f(x)(0)^{[\delta]} = f(x)^{[\delta]}(\bar{\delta}(1), 0) = q_{\beta, x}(\bar{\delta}(1), 0) =$  the smallest  $y$ , for which  $\beta \Vdash \psi(x, y)$ . Then  $f(x)(0)^{[\delta]} = y$  and  $\beta \Vdash \psi(x, y)$ .

So  $\delta \Vdash \psi(x, f(x)(0)^{[\delta]})$  and by Lemma 6.2,  $\delta \Vdash \psi(x, f(x)(0))$ . Since  $\delta \in S'_2$ , there is  $\gamma \in S_2$  with  $\bar{\gamma}(2) = \bar{\delta}(2)$  and by (7),  $\gamma \Vdash \psi(x, f(x)(0))$ . This completes the proof for case  $m = 2$ .

CASE  $m = 1$  is similar with  $f \in a_1$  defined by:

$$f(u, x) = \begin{cases} 0 & \text{if } lh(u) \geq lh(\alpha) \& \neg(u \preceq_0 \bar{\alpha}(1)), \\ \text{the smallest number } y, \text{ for which } \beta \Vdash \psi(x, y) & \\ \text{for the unique } \beta \text{ with } shortest(extension(u), \beta, x) & \\ \text{if } u \preceq_0 \bar{\alpha}(1) \& \exists z [extension(u) \Vdash \psi(x, z)], & \\ \text{undefined} & \text{otherwise.} \end{cases}$$

CASE  $m \geq 3$ .

We use the following functions defined for fixed  $\beta$  and  $x \in \omega$ .

$q_{0, \beta, x} = q_{\beta, x}$  as defined in case  $m = 2$ ;

$$q_{1, \beta, x}(u, k) = \begin{cases} \widehat{K}^1 & \text{if } lh(u) \geq lh(\beta) \& \neg(u \preceq_1 \bar{\beta}(2)), \\ q_{0, \beta, x} & \text{if } u \preceq_1 \bar{\beta}(2), \\ \text{undefined} & \text{otherwise;} \end{cases}$$

$$q_{2,\beta,x}(u,k) = \begin{cases} \hat{K}^2 & \text{if } lh(u) \geq lh(\beta) \& \neg(u \preceq_2 \bar{\beta}(3)), \\ q_{1,\beta,x} & \text{if } u \preceq_2 \bar{\beta}(3), \\ \text{undefined} & \text{otherwise;} \\ \dots & \end{cases}$$

$$q_{m-2,\beta,x}(u,k) = \begin{cases} \hat{K}^{m-2} & \text{if } lh(u) \geq lh(\beta) \& \neg(u \preceq_{m-2} \bar{\beta}(m-1)), \\ q_{m-3,\beta,x} & \text{if } u \preceq_{m-2} \bar{\beta}(m-1), \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Finally, we define function  $f$ :

$$f(u,x) = \begin{cases} \hat{K}^{m-1} & \text{if } lh(u) \geq lh(\alpha) \& \neg(u \preceq_{m-1} \bar{\alpha}(m)), \\ q_{m-2,\beta,x} & \text{for the unique } \beta \text{ with } shortest(extension(u), \beta, x) \\ & \text{if } u \preceq_{m-1} \bar{\alpha}(m) \& \exists y [extension(u) \Vdash \psi(x, y)], \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then  $f \in a_m$  and  $\forall x [\alpha' \Vdash \psi(x, f(x) \underbrace{(0) \dots (0)}_{m-1})]$ .

#### PROOF for $(C2_s)$

It involves similar steps as the proof for  $(C1_s)$  but with new notations.

Denote  $\psi(x, G^n)$  the formula  $\varphi(x, G^n)$ , in which all parameters, except  $x$  and  $G$ , are replaced by objects from corresponding domains.

Denote  $shortest(\alpha, \beta, x) \equiv \alpha \preceq \beta \& (\exists g \in a_n) [\beta \Vdash \psi(x, g)] \& (\forall \gamma \succ \beta) \neg (\exists h \in a_n) [\gamma \Vdash \psi(x, h)]$ .

Clearly,

$$(12) \quad (\exists g \in a_n) [\alpha \Vdash \psi(x, g)] \Rightarrow \exists! \beta shorttest(\alpha, \beta, x).$$

Fix  $\alpha \in M$ . Suppose

$$\alpha \Vdash \forall x \exists! G^n \psi(x, G).$$

Denote  $\alpha' = extension(\bar{\alpha}(m))$ . So by (7),

$$(13) \quad \alpha' \Vdash \forall x \exists! G^n \psi(x, G).$$

By (13), for any  $\beta$  and  $x \in \omega$ :

$$(14) \quad (\beta \preceq \alpha' \vee \alpha' \preceq \beta) \& (g \in a_n) \& [\beta \Vdash \psi(x, g)] \Rightarrow \text{this } g \text{ is unique.}$$

CASE  $m = n + 2$ . First we construct the proof for this case because it clearly demonstrates the main construction. Other cases will be considered later.

Thus,  $sort(\varphi) \leq n + 2$ .

By (7), it is sufficient to prove:

$$(15) \quad \alpha' \Vdash \exists F^m \forall x \psi(x, F(x)(0)).$$

For fixed  $x \in \omega$  and fixed  $\beta$  with  $(\beta \preceq \alpha' \vee \alpha' \preceq \beta)$  by (14) there is a unique function  $Q_{\beta,x}$  such that:

$$Q_{\beta,x}(v,k) = \begin{cases} \hat{K}^n & \text{if } lh(v) \geq lh(\beta) \& \neg(v \preceq_n \bar{\beta}(n+1)), \\ \text{the unique } g \in a_n \text{ with } \beta \Vdash \psi(x,g) & \\ \text{if } v \preceq_n \bar{\beta}(n+1) \& (\exists h \in a_n) [\beta \Vdash \psi(x,h)], & \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Next we construct a function  $f$  that can be used in (15).

$$f(u,x) = \begin{cases} \hat{K}^{n+1} & \text{if } lh(u) \geq lh(\alpha) \& \neg(u \preceq_{n+1} \bar{\alpha}(n+2)), \\ Q_{\beta,x} \text{ for the unique } \beta \text{ with } shortest(extension(u), \beta, x) & \\ \text{if } u \preceq_{n+1} \bar{\alpha}(n+2) \& (\exists g \in a_n) [extension(u) \Vdash \psi(x,g)], & \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Proof that  $f \in a_{n+2}$ .

1) First we prove that  $f : (d_{n+1} \times \omega) \dashrightarrow a_{n+1}$ . Suppose  $f(u,x) \downarrow$ . We need to prove that  $f(u,x) \in a_{n+1}$ .

When  $f(u,x) = \hat{K}^{n+1}$ , it is obvious.

Assume  $f(u,x) = Q_{\beta,x}$ . Then  $u \preceq_{n+1} \bar{\alpha}(n+2)$ ,  $shortest(extension(u), \beta, x)$  and

$$(16) \quad (\exists h \in a_n) [\beta \Vdash \psi(x,h)].$$

Since  $extension(u) \preceq \alpha'$  and  $extension(u) \preceq \beta$ , we have:

$$\beta \preceq \alpha' \vee \alpha' \preceq \beta.$$

The proof that  $Q_{\beta,x} \in a_{n+1}$  consists of three parts.

a) Clearly,  $Q_{\beta,x} : (d_n \times \omega) \dashrightarrow a_n$ .

b) Monotonicity of  $Q_{\beta,x}$ .

Suppose  $Q_{\beta,x}(v,k) \downarrow$  and  $w \preceq_n v$ . Then  $lh(v) \geq lh(\beta)$ . There are two cases.

i)  $\neg(v \preceq_n \bar{\beta}(n+1))$ .

Then  $\neg(w \preceq_n \bar{\beta}(n+1))$  and  $Q_{\beta,x}(w,k) = \hat{K}^n = Q_{\beta,x}(v,k)$ .

ii)  $v \preceq_n \bar{\beta}(n+1)$ .

Then  $w \preceq_n \bar{\beta}(n+1)$ . By (16) and (14) both  $Q_{\beta,x}(w,k)$  and  $Q_{\beta,x}(v,k)$  equal the unique  $g \in a_n$ , for which  $\beta \Vdash \psi(x,g)$ .

c) Completeness of  $Q_{\beta,x}$ .

Consider a path  $S_3$  in  $d_n$  and  $k \in \omega$ . Take  $v \in S_3$  with length  $lh(\beta)$ . There are two cases.

i)  $\bar{\beta}(n+1) \notin S_3$ .

Then  $\neg(v \preceq_n \bar{\beta}(n+1))$  and  $Q_{\beta,x}(v,k) = \hat{K}^n$ .

ii)  $\bar{\beta}(n+1) \in S_3$ .

Then  $v = \bar{\beta}(n+1)$  and by (14), (16),  $Q_{\beta,x}(v,k)$  equals the unique  $g \in a_n$  with  $\beta \Vdash \psi(x,g)$ .

This completes the proof that  $Q_{\beta,x} \in a_{n+1}$  and  $f : (d_{n+1} \times \omega) \dashrightarrow a_{n+1}$ .

2) Monotonicity of  $f$ .

Suppose  $f(u,x) \downarrow$  and  $v \preceq_{n+1} u$ . Then  $lh(v) \geq lh(u) \geq lh(\alpha)$ . There are two cases.

i)  $\neg(u \preceq_{n+1} \bar{\alpha}(n+2))$ .

In this case  $\neg(v \preceq_{n+1} \bar{\alpha}(n+2))$ , so  $f(v, x) = \widehat{K}^{n+1} = f(u, x)$ .

ii)  $u \preceq_{n+1} \bar{\alpha}(n+2)$ .

Then  $v \preceq_{n+1} \bar{\alpha}(n+2)$ ; there is a unique  $\beta$  with  $\text{shortest}(\text{extension}(u), \beta, x)$  and  $f(u, x) = Q_{\beta, x}$ .

For some  $g \in a_n$  we have:  $\text{extension}(u) \Vdash \psi(x, g)$ . Since  $\text{extension}(u) \preceq \text{extension}(v)$ , we have:  $\text{shortest}(\text{extension}(v), \beta, x)$  and by monotonicity of forcing,  $\text{extension}(v) \Vdash \psi(x, g)$ . So  $f(v, x) = Q_{\beta, x}$  for the same  $\beta$ .

3) Completeness of  $f$ .

Consider  $x \in \omega$  and a path  $S_4$  in  $d_{n+1}$ . There are two cases.

i)  $\bar{\alpha}(n+2) \notin S_4$ .

Take  $u \in S_4$  with length  $lh(\alpha)$ . Then  $\neg(u \preceq_{n+1} \bar{\alpha}(n+2))$  and  $f(u, x) = \widehat{K}^{n+1}$ .

ii)  $\bar{\alpha}(n+2) \in S_4$ .

Denote  $S'_4 = \{\text{extension}(v) \mid v \in S_4\}$ . Then  $S'_4$  is a path in  $M$  through  $\alpha'$  and by (13), for some  $\delta \in S'_4$ ,  $(\exists g \in a_n) [\delta \Vdash \psi(x, g)]$ .

Take  $\gamma = \min\{\alpha', \delta\}$  and  $u = \bar{\gamma}(n+2)$ . Then  $u \preceq_{n+1} \bar{\alpha}(n+2)$ ,  $\gamma = \text{extension}(u)$ ,  $u \in S_4$  and  $(\exists g \in a_n) [\gamma \Vdash \psi(x, g)]$ . By (12), there is a unique  $\beta$  with  $\text{shortest}(\gamma, \beta, x)$ . Therefore  $f(u, x) = Q_{\beta, x}$ . This completes the proof of  $f \in a_{n+2}$ .

Proof of (15).

It is sufficient to prove:

$$\forall x [\alpha' \Vdash \psi(x, f(x)(0))].$$

Fix  $x \in \omega$ . By Lemma 2.1.2, it is sufficient to prove:

$$(\forall \text{ path } S_5 \text{ through } \alpha') (\exists \gamma \in S_5) [\gamma \Vdash \psi(x, f(x)(0))].$$

Consider a path  $S_5$  through  $\alpha'$ . Denote  $S'_5 = \{\text{extension}(\bar{\sigma}(n+2)) \mid \sigma \in S_5\}$ . Then  $S'_5$  is also a path in  $M$  through  $\alpha'$ .

By (13), for some  $\delta' \in S'_5$ ,  $(\exists g \in a_n) [\delta' \Vdash \psi(x, g)]$ . Denote  $\delta = \min\{\alpha', \delta'\}$  and  $u = \bar{\delta}(n+2)$ . Then  $\delta \in S'_5$ ,  $u \preceq_{n+1} \bar{\alpha}(n+2)$ ,  $\delta = \text{extension}(u)$  and  $(\exists g \in a_n) [\delta \Vdash \psi(x, g)]$ .

By (12), for some  $\beta$  we have:  $\text{shortest}(\delta, \beta, x)$ ,  $\delta \preceq \beta$  and  $f(x)^{[\delta]} = f(u, x) = Q_{\beta, x}$ . Since  $\delta \preceq \alpha'$  and  $\delta \preceq \beta$ , we have:

$$\beta \preceq \alpha' \vee \alpha' \preceq \beta.$$

Therefore  $f(x)(0)^{[\delta]} = f(x)^{[\delta]}(\bar{\delta}(n+1), 0) = Q_{\beta, x}(\bar{\delta}(n+1), 0) =$  the unique  $g \in a_n$ , for which  $\beta \Vdash \psi(x, g)$ , according to (14). Then  $f(x)(0)^{[\delta]} = g$  and  $\beta \Vdash \psi(x, g)$ .

So  $\delta \Vdash \psi(x, f(x)(0)^{[\delta]})$  and by Lemma 6.2,  $\delta \Vdash \psi(x, f(x)(0))$ . Since  $\delta \in S'_5$ , there is  $\gamma \in S_5$  with  $\bar{\gamma}(n+2) = \bar{\delta}(n+2)$  and by (7),  $\gamma \Vdash \psi(x, f(x)(0))$ . This completes the proof for case  $m = n+2$ .



CASE  $m = n + 1$  is similar with  $f \in a_{n+1}$  defined by:

$$f(u, x) = \begin{cases} \widehat{K}^n & \text{if } lh(u) \geq lh(\alpha) \& \neg(u \preceq_n \bar{\alpha}(n+1)), \\ \text{the unique } g \in a_n, & \text{for which } \beta \Vdash \psi(x, g) \\ & \text{for the unique } \beta \text{ with } shortest(extension(u), \beta, x) \\ & \text{if } u \preceq_n \bar{\alpha}(n+1) \& (\exists h \in a_n) [extension(u) \Vdash \psi(x, h)], \\ \text{undefined} & \text{otherwise.} \end{cases}$$

CASE  $m \geq n + 3$ .

The following functions are defined for fixed  $x \in \omega$  and fixed  $\beta \in M$  with  $(\beta \preceq \alpha' \vee \alpha' \preceq \beta)$ .

$Q_{n,\beta,x} = Q_{\beta,x}$  as defined in case  $m = n + 2$ .

$$Q_{n+1,\beta,x}(u, k) = \begin{cases} \widehat{K}^{n+1} & \text{if } lh(u) \geq lh(\beta) \& \neg(u \preceq_{n+1} \bar{\beta}(n+2)), \\ Q_{n,\beta,x} & \text{if } u \preceq_{n+1} \bar{\beta}(n+2), \\ \text{undefined} & \text{otherwise;} \end{cases}$$

$$Q_{n+2,\beta,x}(u, k) = \begin{cases} \widehat{K}^{n+2} & \text{if } lh(u) \geq lh(\beta) \& \neg(u \preceq_{n+2} \bar{\beta}(n+3)), \\ Q_{n+1,\beta,x} & \text{if } u \preceq_{n+2} \bar{\beta}(n+3), \\ \text{undefined} & \text{otherwise;} \end{cases}$$

...

$$Q_{m-2,\beta,x}(u, k) = \begin{cases} \widehat{K}^{m-2} & \text{if } lh(u) \geq lh(\beta) \& \neg(u \preceq_{m-2} \bar{\beta}(m-1)), \\ Q_{m-3,\beta,x} & \text{if } u \preceq_{m-2} \bar{\beta}(m-1), \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Finally, we define function  $f$ :

$$f(u, x) = \begin{cases} \widehat{K}^{m-1} & \text{if } lh(u) \geq lh(\alpha) \& \neg(u \preceq_{m-1} \bar{\alpha}(m)), \\ Q_{m-2,\beta,x} & \text{for the unique } \beta \text{ with } shortest(extension(u), \beta, x) \\ & \text{if } u \preceq_{m-1} \bar{\alpha}(m) \& (\exists g \in a_n) [extension(u) \Vdash \psi(x, g)], \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then  $f \in a_m$  and  $\forall x [\alpha' \Vdash \psi(x, f(x) \underbrace{(0) \dots (0)}_{m-n-1})]$ . ⊥

**§10. Kripke's schema in the model  $\mathcal{B}_s$ .** In  $LP$  we consider the following Kripke's schema:

$$(KS) \quad \exists G^m [\varphi \equiv \exists x (G(x)(0)^{m-1} \neq 0)],$$

where  $\varphi$  is a formula of  $L$ ,  $G^m$  is not a parameter of  $\varphi$  and  $m \geq \max(\text{sort}(\varphi), 1)$ .

**THEOREM 10.1.** 1.  $LP_s + (C_s) \vdash (KS_s)$ .

2.  $LP + (C) \vdash (KS)$ .

3.  $\mathcal{B}_s \Vdash (KS_s)$ .

PROOF. 1. Axiom  $(CS1_s)$  implies:

$$\forall x \exists y [y \neq 0 \equiv (\vdash_x \varphi)].$$

Since  $sort(\vdash_x \varphi) = sort(\varphi)$ , then by axiom  $(C1_s)$  there exists  $G^m$  such that:

$$\forall x [(G(x)(0)^{m-1} \neq 0) \equiv (\vdash_x \varphi)].$$

Then

$$\exists x [G(x)(0)^{m-1} \neq 0] \equiv \exists x (\vdash_x \varphi)$$

and by axiom  $(CS3_s)$

$$\exists x [G(x)(0)^{m-1} \neq 0] \equiv \varphi.$$

2. Follows from part 1.

3. It follows from part 1, Theorem 9.1 and the soundness of the model  $\mathcal{B}_s$  for  $LP_s$ .  $\dashv$

## §11. Other intuitionistic principles in the model $\mathcal{B}_s$ .

### 11.1. Weak continuity.

$$(WC) \quad \forall \mathcal{F}^n \exists x \varphi(\mathcal{F}, x) \supset \forall \mathcal{F}^n \exists x \exists y \forall \mathcal{G}^n [\bar{\mathcal{G}}(y) = \bar{\mathcal{F}}(y) \supset \varphi(\mathcal{G}, x)],$$

where  $\varphi$  is a formula of  $LP$  with  $sort(\varphi) \leq n$  and  $\varphi$  does not have non-lawlike parameters of type  $n$  other than  $\mathcal{F}^n$ .

THEOREM 11.1. 1.  $LP_s + (LL3_s) \vdash (WC_s)$ .

2.  $LP + (LL3) \vdash (WC)$ .

3.  $\mathcal{B}_s \Vdash (WC_s)$ .

PROOF. 1. Denote  $\psi(x, y, \mathcal{F}^n, \mathcal{G}^n)$  the formula  $\bar{\mathcal{G}}(y) = \bar{\mathcal{F}}(y) \supset \varphi(\mathcal{G}, x)$ .

By  $(LL3_s)$ ,

$$\forall \mathcal{F}^n \forall x [\varphi(\mathcal{F}, x) \supset \exists y \forall \mathcal{G}^n \psi(x, y, \mathcal{F}, \mathcal{G})].$$

Then

$$\forall \mathcal{F}^n [\exists x \varphi(\mathcal{F}, x) \supset \exists x \exists y \forall \mathcal{G}^n \psi(x, y, \mathcal{F}, \mathcal{G})]$$

and

$$\forall \mathcal{F}^n \exists x \varphi(\mathcal{F}, x) \supset \forall \mathcal{F}^n \exists x \exists y \forall \mathcal{G}^n \psi(x, y, \mathcal{F}, \mathcal{G}).$$

2. Follows from part 1.

3. It follows from part 1, Theorem 8.1 and the soundness of the model  $\mathcal{B}_s$  for  $LP_s$ .  $\dashv$

### 11.2. Bar induction.

$$(BI) \quad \forall F^1 \exists x \varphi(\bar{F}(x)) \wedge \forall x \forall y [\varphi(y) \supset \varphi(y * \langle x \rangle)]$$

$$\wedge \forall y [\forall x \psi(y * \langle x \rangle) \supset \psi(x)] \wedge \forall y [\varphi(y) \supset \psi(y)] \supset \psi(0).$$

THEOREM 11.2.  $\mathcal{B}_s \Vdash (BI_s)$ .

PROOF. The proof is the same as for Van Dalen's model [12].  $\dashv$

**11.3. Markov principle.** This principle is often considered in constructive mathematics.

Markov Principle:

$$(MP) \forall x [\varphi(x) \vee \neg\varphi(x)] \wedge \neg\neg\exists x\varphi(x) \supset \exists x\varphi(x).$$

Markov Rule:

$$(MR) \text{ If } \forall x [\varphi(x) \vee \neg\varphi(x)] \text{ and } \neg\neg\exists x\varphi(x), \text{ then } \exists x\varphi(x).$$

**THEOREM 11.3.** 1. *The Markov Rule (MR) does not hold in  $\mathcal{B}_s$ .*  
2. *The Markov Principle (MP) does not hold in  $\mathcal{B}_s$ .*

**PROOF.** 1. Denote  $\psi(x, F^1) \Leftrightarrow (\exists k < x)(F(k) > 0)$ . For  $u \in d_0, n \in \omega$ , define:

$$f(u, n) = \begin{cases} \langle \langle u \rangle_0 \rangle_n & \text{if } n < lh(u), \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then  $f \in a_1$ . Denote  $\varphi(x) \Leftrightarrow \psi(x, f)$ . We need to check that the premises of (MR) are forced in the model:

$$(17) \quad \mathcal{B}_s \Vdash \forall x [\varphi(x) \vee \neg\varphi(x)],$$

$$(18) \quad \mathcal{B}_s \Vdash \neg\neg\exists x\varphi(x)$$

but the conclusion is not forced:

$$(19) \quad \neg(\mathcal{B}_s \Vdash \exists x\varphi(x)).$$

Proof of (17). We need to prove:

$$(\forall x \in \omega)(\forall \text{ path } S \text{ in } M)(\exists \alpha \in S) [\alpha \Vdash \varphi(x) \text{ or } \alpha \Vdash \neg\varphi(x)].$$

Fix  $x \in \omega$  and a path  $S$  in  $M$ . Choose  $\alpha \in S$  with  $lh(\alpha) = x$ . Then for any  $n < x, f^{[\alpha]}(n) \downarrow$ .

If  $(\exists n < x)(f^{[\alpha]}(n) > 0)$ , then  $\alpha \Vdash \varphi(x)$ . Otherwise  $(\forall n < x)(f^{[\alpha]}(n) = 0)$ , which implies  $\alpha \Vdash \neg\varphi(x)$ .

Proof of (18). It is sufficient to prove:

$$\forall \alpha \exists x (\exists y < x)(\exists \beta \preceq \alpha)(\beta \Vdash f(y) > 0).$$

Fix  $\alpha \in M$ . Denote  $n = lh(\alpha)$  and choose

$$x = n + 1, y = n, \beta = \langle \langle \alpha \rangle_0 * \langle 1 \rangle, \langle \alpha \rangle_1 * \langle \hat{K}^1 \rangle, \dots, \langle \alpha \rangle_{s-1} * \langle \hat{K}^{s-1} \rangle \rangle.$$

Then  $\beta \preceq \alpha, f^{[\beta]}(y) = \langle \langle \beta \rangle_0 \rangle_n = 1$  and  $\beta \Vdash f(y) > 0$ .

Proof of (19). For any  $n \geq 0$  we define a function  $g_n : \omega \rightarrow a_n$  as follows:  $g_0 = \lambda m.0$  and for  $n \geq 1, g_n = \lambda m.\hat{K}^n$ . We take

$$S = \{ \langle \bar{g}_0(m), \dots, \bar{g}_{s-1}(m) \rangle \mid m \in \omega \}.$$

Then  $S$  is a path in  $M$ . For any  $\alpha \in S$  and  $x = lh(\alpha)$  we have:

$$\text{for any } y < x, f^{[\alpha]}(y) = f(\bar{\alpha}(1), y) = \langle \langle \alpha \rangle_0 \rangle_y = g_0(y) = 0.$$

This implies (19).

2. Follows from part 1.

◄

#### 11.4. Formal Church thesis.

$$(CT) \quad \forall x \exists y \varphi(x, y) \supset \exists e \forall x \exists y [\{e\}(x) = y \wedge \varphi(x, y)].$$

THEOREM 11.4.  $LP + (C1) + (CT)$  is inconsistent.

PROOF. Van Dalen [12] showed that  $HA + (C1) + (CT)$  is inconsistent, which implies the theorem.  $\dashv$

**§12. Intuitionistic theory  $SLP$  and classical theory  $TI$ .** By using reverse mathematics approach we combine in one axiomatic theory all intuitionistic principles that were shown to hold in the model  $\mathcal{B}_s$ :

$$SLP = LP + (LL) + (C) + (BI).$$

In the name of the theory  $SLP$ ,  $S$  stands for strong,  $L$  for lawless and  $P$  for proof (relating to the "creating subject"). The proof-theoretical strength of  $SLP$  will be investigated in section 14; in particular we will show that  $SLP$  is stronger than the second order arithmetic. In this section we compare  $SLP$  with a classical theory  $TI$ .

**12.1. Theories  $TI$  and  $TI_s$ .** Theory  $TI$  is a subsystem of typed set theory with arithmetic at the bottom level. The language of  $TI$  has variables of type  $n$ :  $x^n, y^n, z^n, \dots$  for  $n = 0, 1, 2, \dots$ , a constant  $0$  of type  $0$ , functional symbols  $S, +, \cdot$ , and predicate symbols  $=_n$  and  $\in_n$  ( $n \geq 0$ ).

Terms are defined recursively as follows.

1. Any variable of type  $n$  is a term of type  $n$ .
2. The constant  $0$  is a term of type  $0$ .
3. If  $t$  and  $\tau$  are terms of type  $0$ , then  $St, t + \tau$ , and  $t \cdot \tau$  are terms of type  $0$ .

Atomic formulas:

$$\begin{aligned} t =_n r, \\ t \in_n \tau, \text{ where } t \text{ and } r \text{ are terms of type } n, \text{ and } \tau \text{ is a term of type } n + 1. \end{aligned}$$

Formulas are constructed from atomic formulas using logical connectives and quantifiers.

For a formula  $\varphi$ , its sort  $srt(\varphi)$  is the maximal type of parameters in  $\varphi$ .

The theory  $TI$  has classical predicate logic  $CPC$  with equality axioms and the following non-logical axioms.

1.  $Sx^0 \neq 0$ ,  $Sx^0 = Sy^0 \supset x = y$ .
2.  $x^0 + 0 = x$ ,  $x^0 + Sy^0 = S(x + y)$ .
3.  $x^0 \cdot 0 = 0$ ,  $x^0 \cdot Sy^0 = x \cdot y + x$ .
4. Induction for natural numbers:  $\varphi(0) \wedge \forall x^0 (\varphi(x) \supset \varphi(Sx)) \supset \forall x^0 \varphi(x)$ .
5. Comprehension axiom:  $\exists x^{n+1} \forall z^n (z \in x \equiv \varphi(z))$ , where  $srt(\varphi) \leq n + 1$  and  $x^{n+1}$  is not a parameter of  $\varphi$ .
6. Extensionality axiom:  $\forall z^n (z \in x^{n+1} \equiv z \in y^{n+1}) \supset x = y$ .

McNaughton [9] considered two typed set theories  $T$  and  $I$ , which differ from  $TI$  only in the comprehension axiom: in  $T$  this axiom is stated without restrictions on the formula  $\varphi$  and in  $I$  the formula  $\varphi$  in the comprehension axiom has no types greater than  $n + 1$ .

Thus, all three theories  $T$ ,  $TI$  and  $I$  have impredicative comprehension axioms; this axiom is the least predicative in  $T$  and most predicative in  $I$ .

In [9] the variables in  $T$  and  $I$  have types  $s = 1, 2, 3, \dots$  with type 1 at the bottom level. Here we change them to  $s = 0, 1, 2, \dots$  with type 0 at the bottom level to make the notations consistent with ours.

We denote  $T_s$ ,  $TI_s$  and  $I_s$  the fragments of  $T$ ,  $TI$  and  $I$ , respectively, in which the types of variables are not greater than  $s$  ( $s = 0, 1, 2, \dots$ ).

**12.2. Interpretation of  $TI$  in  $TI$  without extensionality.** First we define a formula  $x^n \approx_n y^n$  of the language  $TI$  by induction on  $n$ .

$x^0 \approx_0 y^0$  is  $x = y$ ;

$x^{n+1} \approx_{n+1} y^{n+1}$  is  $(\forall z^n \in x)(\exists u^n \in y)(z \approx_n u) \wedge (\forall z^n \in y)(\exists u^n \in x)(z \approx_n u)$ .

Clearly,  $\approx_n$  is an equivalence relation.

For any formula  $\varphi$  of  $TI$  we define  $\varphi^*$  by induction on the complexity of  $\varphi$ .

$(t =_n \tau)^*$  is  $t \approx_n \tau$ ;

$(t \in_n \tau)^*$  is  $(\exists z^n \in \tau)(z \approx_n t)$ ;

$(\perp)^*$  is  $\perp$ ;

$(\psi \theta \chi)^*$  is  $\psi^* \theta \chi^*$ , where  $\theta$  is a logical connective  $\wedge, \vee$  or  $\supset$ ;

$(Qx^n \psi)^*$  is  $Qx^n \psi^*$ , where  $Q$  is a quantifier  $\forall$  or  $\exists$ .

Denote  $TI_s^*$  and  $TI^*$  the theories  $TI_s$  and  $TI$ , respectively, without the extensionality axiom.

LEMMA 12.1.

$$TI_s \vdash \varphi \Rightarrow TI_s^* \vdash \bar{\varphi}^*,$$

where  $\bar{\varphi}$  is a closure of formula  $\varphi$ .

PROOF. The proof is by induction on the length of the derivation of  $\varphi$ .

For logical axioms and equality axioms the proof is obvious from the definition of  $\varphi^*$ . Therefore

$$(20) \quad CPC_s^+ \vdash \varphi \Rightarrow CPC_s^+ \vdash \varphi^*,$$

where  $CPC_s^+$  is the classical predicate calculus with equality in the language of  $TI_s$ .

Of non-logical axioms we will consider the only non-trivial case of comprehension axiom:

$$\exists x^{n+1} \forall z^n (z \in x \equiv \psi(z)), \text{ where } srt(\psi) \leq n+1.$$

Since  $CPC_s^+ \vdash z^n =_n u^n \wedge \psi(z) \supset \psi(u)$ , by (20) we have:

$$CPC_s^+ \vdash z^n \approx_n u^n \wedge \psi^*(u) \supset \psi^*(z).$$

Clearly,  $srt(\psi^*) = srt(\psi)$ , so by the comprehension axiom in  $TI_s^*$ , there exists  $x^{n+1}$  such that

$$\forall z^n [z \in x \equiv \psi^*(z)].$$

It remains to prove:

$$\forall z^n [(z \in x)^* \equiv \psi^*(z)].$$

If  $(z \in x)^*$ , then for some  $u^n \in x, z \approx_n u$ . So  $\psi^*(u)$  and by (20),  $\psi^*(z)$ . The part  $\psi^*(z) \supset (z \in x)^*$  is obvious.  $\dashv$

**12.3. Interpretation of  $TI^*$  in  $SLP$ .** First for each term  $t$  of the language  $TI$  we define a term  $t'$  of the language  $SLP$ . If  $t$  is a numerical term, then  $t'$  is obtained from  $t$  by replacing each variable  $x_i^0$  by  $x_i$ . For  $n \geq 1, (x_i^n)' = F_i^n$ .

Next for each formula  $\varphi$  of  $TI$  we define a formula  $\varphi'$  of  $SLP$  by induction on the complexity of  $\varphi$ .

$(t =_n \tau)'$  is  $t' =_n \tau'$ ;

$(t \in_0 \tau)'$  is  $\exists y [\tau'(y) = S(t')]$ ;

for  $n \geq 1, (t \in_n \tau)'$  is  $\exists y [\tau'(y) = N^n(t')]$ ;

$\perp'$  is  $\perp$ ;

$(\psi \theta \chi)'$  is  $\psi' \theta \chi'$ , where  $\theta$  is a logical connective  $\wedge, \vee$  or  $\supset$ ;

$(Qx^n \psi)'$  is  $Q(x^n)' \psi'$ , where  $Q$  is a quantifier  $\forall$  or  $\exists$ .

Finally, for each formula  $\varphi$  of  $SLP$  we define a formula  $\varphi^-$  of  $SLP$  by induction on the complexity of  $\varphi$ .

If  $\varphi$  is an atomic formula, then  $\varphi^-$  is  $\neg \neg \varphi$ ;

$\perp^-$  is  $\perp$ ;

$(\psi \theta \chi)^-$  is  $\psi^- \theta \chi^-$ , where  $\theta$  is a logical connective  $\wedge$  or  $\supset$ ;

$(\psi \vee \chi)^-$  is  $\neg(\neg \psi^- \wedge \neg \chi^-)$ ;

$(\forall X \psi)^-$  is  $\forall X \psi^-$  and

$(\exists X \psi)^-$  is  $\neg \forall X \neg \psi^-$ , where  $X$  is any variable of  $SLP$ .

LEMMA 12.2.

$$TI_s^* \vdash \varphi \Rightarrow SLP_s \vdash (\bar{\varphi})^-.$$

PROOF. Proof is similar to the proof in [6].  $\dashv$

Denote  $int(\varphi) = ((\varphi^*)')^-$ .

THEOREM 12.1. 1.  $TI_s \vdash \varphi \Rightarrow SLP_s \vdash int(\bar{\varphi})$ .

2.  $TI \vdash \varphi \Rightarrow SLP \vdash int(\bar{\varphi})$ .

PROOF. 1. It follows from Lemmas 12.1 and 12.2.

2. Follows from part 1.  $\dashv$

Formulas of the first order arithmetic can be considered the same in  $TI$  and  $SLP$  by identifying each variable  $x_i^0$  of  $TI$  with the variable  $x_i$  of  $SLP$ .

LEMMA 12.3. Suppose  $\varphi$  is a closed formula of first order arithmetic of the form  $\forall n \psi(n)$ , where formula  $\psi$  defines a primitive recursive predicate. Then  $SLP_0 \vdash int(\varphi) \equiv \varphi$ .

PROOF. Since  $\psi$  is an arithmetic formula, then  $\psi^* = \psi$  and  $\psi' = \psi$ . Since  $\psi$  defines a primitive recursive predicate, then  $\psi^- \equiv \psi$ . Therefore  $int(\varphi) \equiv \forall n int(\psi) \equiv \forall n \psi \equiv \varphi$ .  $\dashv$

**§13. The relative strengths of theories  $TI_s$  and  $TI$ .** In [1] and [2] Bernini intended to create a strong intuitionistic theory of functionals but the theory in [1] happened to be inconsistent and the consistency of its modification in [2] was not proved. We proved the consistency of our intuitionistic theory  $SLP$  of functionals by constructing the model  $\mathcal{B}_s$ . Now we are interested in the strength of  $SLP$ . We will show that it is the same as the strength of the classical theory  $TI$ . We will also investigate the strengths of the fragments  $SLP_s$  and  $TI_s$ .

**13.1. Constructing a truth predicate for  $TI_s$  in  $TI_{s+1}$ .** Fix  $s \geq 1$ . We will construct in  $TI_{s+1}$  a truth predicate  $Tr$  for formulas of the theory  $TI_s$ . In this subsection details of the construction will be described and its applications to the relative strengths of our theories will be given in next sub-sections.

First we introduce some notations in the language  $TI$ . We will often use letters  $i, j, k, l, m, n, p, q, r$  for natural numbers, in particular instead of variables of type 0 in the language  $TI$ .

For a natural number  $k, \check{k} = \underbrace{S \dots S}_k 0$  is a term of the language  $TI$ .

$$\{x^k\}^1 = \{x\}; \{x^k\}^{n+1} = \{\{x\}^n\}.$$

For any  $n = 0, 1, \dots, s$  we define an ordered pair  $[x^n, y^n]$  by induction on  $n$ . An ordered pair  $[x^0, y^0]$  of natural numbers is coded by a natural number; the code is given by a primitive recursive function as usual in the first order arithmetic.

$$[x^{n+1}, y^{n+1}] = \{[\{0\}^n, z^n] \mid z \in x\} \cup \{[\{1\}^n, z^n] \mid z \in y\}.$$

Thus, the ordered pair  $[x^n, y^n]$  of objects of type  $n$  also has type  $n$ .

Next we define a finite sequence  $\langle x_1^n, \dots, x_k^n \rangle$  by induction on  $n$ .

$\langle x_1^0, \dots, x_k^0 \rangle$  is coded by a natural number as usual in the first order arithmetic.

$$\langle x_1^{n+1}, \dots, x_k^{n+1} \rangle = \left\{ [\{0\}^n, \{\check{k}\}^n] \right\} \cup \bigcup_{i=1}^k \{ [\{\check{i}\}^n, z^n] \mid z \in x_i \}.$$

For arbitrary types  $n_1, \dots, n_k$  and  $m = \max\{n_1, \dots, n_k\}$  we define:

$$\langle x_1^{n_1}, \dots, x_k^{n_k} \rangle = \langle \{x_1\}^{m-n_1}, \dots, \{x_k\}^{m-n_k} \rangle.$$

We will fix Gödel numbering of all expressions of the language  $TI_s$ .

For any expression  $g$  we denote  $\perp g \perp$  its Gödel number in this numbering.  $t_n$  denotes the arithmetic term of  $TI_s$  with Gödel number  $n$  and  $\varphi_n$  denotes the formula of  $TI_s$  with Gödel number  $n$ ;  $t_n$  or  $\varphi_n$  is undefined if  $n$  is not the Gödel number of an arithmetic term or a formula, respectively.

Next we introduce a predicate  $Eval_k(f^k), k = 0, 1, \dots, s$ , with the meaning: " $f$  is an evaluation of all variables of type  $k$ ", that is  $f$  defines an infinite sequence of objects, where the  $i$ th element evaluates variable  $x_i^k$  ( $i = 1, 2, \dots$ ). A functional symbol  $Val_k(f, i)$  is defined next; it produces the  $i$ th element of  $f$ .

For  $k = 1, 2, \dots, s$

$$Eval_k(f^k) \Leftrightarrow (\forall x^{k-1} \in f) [x = \emptyset^{k-1} \vee \exists i, z^{k-1} (x = \langle i, z \rangle)];$$

$$Val_k(f^k, i) = \{z^{k-1} \mid \langle i, z \rangle \in f\}.$$

In the special case  $k = 0$  the parameter of the formula  $Eval_0$  has type 1:

$$Eval_0(f^1) \Leftrightarrow (\forall x^0 \in f) \exists i, n (x = \langle i, n \rangle) \wedge \forall i \exists! n (\langle i, n \rangle \in f).$$

Thus,  $Eval_0(f^1)$  means  $f : \omega \rightarrow \omega$ . Clearly,

$$TI_{s+1} \vdash \forall i \exists! n [(Eval_0(f^1) \supset \langle i, n \rangle \in f) \wedge (\neg Eval_0(f^1) \supset n = 0)].$$

So we can introduce a functional symbol  $Val_0(f^1, i)$  such that

$$TI_{s+1} \vdash Eval_0(f^1) \supset \langle i, Val_0(f, i) \rangle \in f.$$

Denote  $\bar{f}$  the list of variables  $f_0^1, f_1^1, f_2^2, \dots, f_s^s$  and

$$Eval(\bar{f}) \Leftrightarrow \bigwedge_{k=0}^s Eval_k(f_k).$$

For  $k = 1, 2, \dots, s$  denote

$$Subst_k(f^k, i, y^k) = \{ \langle j, z^{k-1} \rangle \in f \mid j \neq i \} \cup \{ \langle i, z^{k-1} \rangle \mid z \in y \}.$$

Then

$$(21) \quad \begin{aligned} TI_{s+1} \vdash Eval_k(f^k) \wedge g^k &= Subst_k(f, i, y^k) \\ &\supset Eval_k(g) \wedge Val_k(g, i) = y \wedge (\forall j \neq i) [Val_k(g, j) = Val_k(f, j)]. \end{aligned}$$

Similarly, denote

$$Subst_0(f^1, i, y^0) = \{ \langle j, z^0 \rangle \in f \mid j \neq i \} \cup \{ \langle i, y \rangle \};$$

then it has a property similar to (21).

Thus,  $Subst_k(f, i, y)$  is the new evaluation obtained from evaluation  $f$  by changing the value for variable  $x_i^k$  to  $y$ .

For any  $k = 0, 1, \dots, s$ , denote  $Sub_k(\bar{f}, i, y^k)$  the list of variables obtained from list  $\bar{f}$  by replacing  $f_k$  with  $Subst_k(f_k, i, y^k)$ .

LEMMA 13.1.  $TI_{s+1} \vdash Eval(\bar{f}) \supset Eval(Sub_k(\bar{f}, i, k)), k = 0, 1, \dots, s.$

PROOF. Follows from the definitions of  $Eval$  and  $Sub_k$ .  $\dashv$

We denote  $Term(n)$  the arithmetic formula stating that  $n$  is the Gödel number of an arithmetic term of  $TI_s$ . We denote  $Subterm(m, n)$  the arithmetic formula stating that  $m$  is the Gödel number of a sub-term of the term with Gödel number  $n$ .

In next few steps we define a functional symbol  $Arval(n, f^1)$  that produces the value of the arithmetic term  $t_n$  under evaluation  $f^1$ .

First we define a predicate  $Arterm(n, f^1, a^1)$  which means that  $a$  is the set of the values of all sub-terms of the term  $t_n$  under evaluation  $f$ .

$$\begin{aligned} Arterm(n, f^1, a^1) &\Leftrightarrow Term(n) \wedge Eval_0(f) \\ &\wedge (\forall x^0 \in a) \exists m, k [x = \langle m, k \rangle \wedge Subterm(m, n)] \\ &\wedge \forall m [Subterm(m, n) \supset B(m, a)], \end{aligned}$$

where  $B(m, a) \Leftrightarrow B_1 \vee B_2 \vee B_3 \vee B_4 \vee B_5$  and the predicates  $B_1 - B_5$  determine the values of sub-terms of  $t_n$  as follows.

$$B_1 \Leftrightarrow m = \perp 0 \perp \wedge \forall k (\langle m, k \rangle \in a \equiv k = 0).$$



$$B_2 \Leftarrow m = \perp x_i^0 \perp \wedge \forall k [\langle m, k \rangle \in a \equiv k = Val_0(f, i)].$$

$$B_3 \Leftarrow \exists j \{m = \perp St_j \perp \wedge \forall k [\langle m, k \rangle \in a \equiv \exists l (\langle j, l \rangle \in a \wedge k = Sl)]\}.$$

Denote  $+$  as  $\theta_4$  and  $\cdot$  as  $\theta_5$ . For  $p = 4, 5$ ,

$$B_p \Leftarrow \exists i, j \{m = \perp t_i \theta_p t_j \perp \wedge \forall k [\langle m, k \rangle \in a \equiv \exists r, l (\langle i, r \rangle \in a \wedge \langle j, l \rangle \in a \wedge k = r \theta_p l)]\}.$$

Denote

$$a_{||m} = \{\langle j, k \rangle \in a \mid Subterm(j, m)\}.$$

LEMMA 13.2. *The following formulas are derived in  $TI_{s+1}$ .*

1.  $Subterm(m, n) \wedge Arterm(n, f^1, a^1) \supset Arterm(m, f, a_{||m})$ .
2.  $Arterm(n, f^1, a^1) \wedge Arterm(n, f, b^1) \supset a = b$ .
3.  $Term(n) \wedge Eval_0(f^1) \supset \exists! a^1 Arterm(n, f, a)$ .

PROOF. 1. Assume the premises. For any  $i$  with  $Subterm(i, m)$ , we have:

$$(22) \quad B(i, a) \text{ and } \forall k (\langle i, k \rangle \in a \equiv \langle i, k \rangle \in a_{||m}).$$

To prove  $Arterm(m, f, a_{||m})$ , it is sufficient to show for any  $j$ :

$$(23) \quad Subterm(j, m) \supset B(j, a_{||m}).$$

When  $t_j$  is a constant or a variable, (23) follows from  $B(j, a)$ .

Suppose  $j = \perp t_p + t_q \perp$ . Since  $B(j, a)$ , we have for any  $k$ :

$$\langle j, k \rangle \in a \equiv \exists r, l (\langle p, r \rangle \in a \wedge \langle q, l \rangle \in a \wedge k = r + l)$$

and by (22):

$$\langle j, k \rangle \in a_{||m} \equiv \exists r, l (\langle p, r \rangle \in a_{||m} \wedge \langle q, l \rangle \in a_{||m} \wedge k = r + l),$$

which implies  $B(j, a_{||m})$ .

For other forms of  $t_j$  the proof is similar.

2. Assume  $Arterm(n, f^1, a^1) \wedge Arterm(n, f, b^1)$ . Then  $Term(n)$  and  $Eval_0(f)$ .

It is sufficient to prove:

$$(24) \quad \forall m [Subterm(m, n) \supset \forall k (\langle m, k \rangle \in a \equiv \langle m, k \rangle \in b)].$$

We will prove it by induction on  $m$ . The cases of constant 0 and variables are obvious.

Assume (24) holds for numbers less than  $m$ . We will consider only the case  $m = \perp t_i + t_j \perp$ , other cases are similar. By the inductive assumption,

$$\forall k (\langle i, k \rangle \in a \equiv \langle i, k \rangle \in b) \text{ and } \forall k (\langle j, k \rangle \in a \equiv \langle j, k \rangle \in b).$$

Therefore for any  $k$ ,

$$\begin{aligned} \langle m, k \rangle \in a &\equiv \exists r, l (\langle i, r \rangle \in a \wedge \langle j, l \rangle \in a \wedge k = r + l) \\ &\equiv \exists r, l (\langle i, r \rangle \in b \wedge \langle j, l \rangle \in b \wedge k = r + l) \equiv \langle m, k \rangle \in b. \end{aligned}$$

3. The uniqueness was proven in part 2. We prove the existence by induction on  $n$ .

- 1) If  $n = \perp 0 \perp$ , we take  $a^1 = \{\langle n, 0 \rangle\}$ .

2) If  $n = \perp x_i^0 \perp$ , we take  $a^1 = \{\langle n, Val_0(f, i) \rangle\}$ .

In each case  $Arterm(n, f, a)$  is obvious.

Assume the existence for numbers less than  $n$ .

3)  $n = \perp t_i + t_j \perp$ .

By the inductive assumption, there exist  $c^1$  and  $d^1$  such that  $Arterm(i, f, c)$  and  $Arterm(j, f, d)$ . We take

$$a^1 = c \cup d \cup \{\langle n, r + l \rangle \mid \langle i, r \rangle \in c \wedge \langle j, l \rangle \in d\}$$

and prove  $Arterm(n, f, a)$ .

First we prove:

$$(25) \quad \forall p [Subterm(p, i) \supset \forall k (\langle p, k \rangle \in a \equiv \langle p, k \rangle \in c)] .$$

Assume  $Subterm(p, i)$ . If  $\neg Subterm(p, j)$ , then the conclusion is obvious. If  $Subterm(p, j)$ , then  $Arterm(p, f, c_{||p})$  and  $Arterm(p, f, d_{||p})$  by part 1,  $c_{||p} = d_{||p}$  by part 2 and for any  $k$ :

$$\langle p, k \rangle \in a \equiv \langle p, k \rangle \in c_{||p} \cup d_{||p} \equiv \langle p, k \rangle \in c .$$

(25) is proven. Similarly,

$$(26) \quad \forall p [Subterm(p, j) \supset \forall k (\langle p, k \rangle \in a \equiv \langle p, k \rangle \in d)] .$$

To prove  $Arterm(n, f, a)$ , it is sufficient to show:

$$(27) \quad \forall m [Subterm(m, n) \supset B(m, a)] .$$

Assume  $Subterm(m, n)$ .

i)  $m = n$ .

Applying (25) to  $p = i$  and (26) to  $p = j$ , we get for any  $k$ :

$$\begin{aligned} \langle n, k \rangle \in a &\equiv \exists r, l (\langle i, r \rangle \in c \wedge \langle j, l \rangle \in d \wedge k = r + l) \\ &\equiv \exists r, l (\langle i, r \rangle \in a \wedge \langle j, l \rangle \in a \wedge k = r + l) , \end{aligned}$$

which implies  $B(n, a)$ .

ii)  $m < n$ .

Then  $Subterm(m, i) \vee Subterm(m, j)$ . We can assume  $Subterm(m, i)$  without loss of generality. Applying (25) to each possible form of  $t_m$ , we get  $B(m, a)$  due to  $Arterm(i, f, c)$ .

For example, if  $t_m = S(t_q)$ , by (25) we have for any  $k$ :

$$\langle m, k \rangle \in a \equiv \langle m, k \rangle \in c \equiv \exists l (\langle q, l \rangle \in c \wedge k = Sl) \equiv \exists l (\langle q, l \rangle \in a \wedge k = Sl) ,$$

which implies  $B(m, a)$ .

This completes the proof for case 3). The cases of other functional symbols are similar.  $\dashv$

LEMMA 13.3. *It is derived in  $TI_{s+1}$  that the formula*

$$Subterm(m, n) \wedge Arterm(n, f^1, a^1)$$

*implies each of the following formulas:*

$$1. \ m = \perp 0 \perp \supset \forall k (\langle m, k \rangle \in a \equiv k = 0) .$$

$$2. \ m = \perp x_i^0 \perp \supset \forall k [\langle m, k \rangle \in a \equiv k = Val_0(f, i)] .$$

3.  $m = \perp S(t_j) \perp \supset \forall k [\langle m, k \rangle \in a \equiv \exists l (\langle j, l \rangle \in a \wedge k = Sl)]$ .
4.  $m = \perp t_i \theta t_j \perp \supset \forall k [\langle m, k \rangle \in a \equiv \exists r, l (\langle i, r \rangle \in a \wedge \langle j, l \rangle \in a \wedge k = r\theta l)]$ ,  
where  $\theta$  denotes  $\cdot$  or  $+$ .

PROOF. Follows from the definition of *Arterm*. ⊢

Next we define a functional symbol

$$Art^1(f^1) = \{ \langle n, k \rangle \mid \exists a^1 (Arterm(n, f, a) \wedge \langle n, k \rangle \in a) \}.$$

LEMMA 13.4.

$$TI_{s+1} \vdash Subterm(m, n) \wedge Arterm(n, f^1, a^1) \supset \forall k [\langle m, k \rangle \in a \equiv \langle m, k \rangle \in Art(f)].$$

PROOF. 1. Assume the premises. By Lemma 13.2.1,  $Arterm(m, f, a_{||m})$ . Fix  $k$ .

$\Rightarrow$  If  $\langle m, k \rangle \in a$ , then  $\langle m, k \rangle \in a_{||m}$  and  $\langle m, k \rangle \in Art(f)$  by the definition of *Art*.

$\Leftarrow$  Suppose  $\langle m, k \rangle \in Art(f)$ . Then there exists  $b^1$  such that  $Arterm(m, f, b)$  and  $\langle m, k \rangle \in b$ . By Lemma 13.2.2,  $b = a_{||m}$ . So  $\langle m, k \rangle \in a_{||m}$  and  $\langle m, k \rangle \in a$ . ⊢

LEMMA 13.5. *It is derived in  $TI_{s+1}$  that the formula*

$$Eval_0(f^1)$$

*implies each of the following formulas.*

1.  $\langle \perp 0 \perp, k \rangle \in Art(f) \equiv k = 0$ .
2.  $\langle \perp x_i^0 \perp, k \rangle \in Art(f) \equiv k = Val_0(f, i)$ .
3.  $\langle \perp S(t_j) \perp, k \rangle \in Art(f) \equiv \exists l (\langle j, l \rangle \in Art(f) \wedge k = Sl)$ .
4.  $\langle \perp t_i \theta t_j \perp, k \rangle \in Art(f) \equiv \exists r, l (\langle i, r \rangle \in Art(f) \wedge \langle j, l \rangle \in Art(f) \wedge k = r\theta l)$ ,  
where  $\theta$  denotes  $\cdot$  or  $+$ .

PROOF. Assume  $Eval_0(f^1)$ . By Lemma 13.2.3, for any  $n$  with  $Term(n)$ , there is a set  $a^1$  such that  $Arterm(n, f, a)$ , and by Lemma 13.4:

$$(28) \quad \forall m [Subterm(m, n) \supset \forall k (\langle m, k \rangle \in a \equiv \langle m, k \rangle \in Art(f))].$$

1. Denote  $n = \perp 0 \perp$ . By (28), we have for any  $k$ :

$$\langle n, k \rangle \in Art(f) \equiv \langle n, k \rangle \in a \equiv k = 0$$

by Lemma 13.3.1.

2. Denote  $n = \perp x_i^0 \perp$ . By (28), we have for any  $k$ :

$$\langle n, k \rangle \in Art(f) \equiv \langle n, k \rangle \in a \equiv k = Val_0(f, i)$$

by Lemma 13.3.2.

Parts 3 is similar to part 4, which we will consider next.

4. Denote  $n = \sqcup t_i + t_j \sqcup$ . By (28):

$$\begin{aligned} \forall k (\langle n, k \rangle \in \text{Art}(f) &\equiv \langle n, k \rangle \in a); \\ \forall k (\langle i, k \rangle \in \text{Art}(f) &\equiv \langle i, k \rangle \in a); \\ \forall k (\langle j, k \rangle \in \text{Art}(f) &\equiv \langle j, k \rangle \in a). \end{aligned}$$

By Lemma 13.3.4, for any  $k$ :

$$\begin{aligned} \langle n, k \rangle \in \text{Art}(f) &\equiv \langle n, k \rangle \in a \equiv \exists r, l (\langle i, r \rangle \in a \wedge \langle j, l \rangle \in a \wedge k = r + l) \\ &\equiv \exists r, l (\langle i, r \rangle \in \text{Art}(f) \wedge \langle j, l \rangle \in \text{Art}(f) \wedge k = r + l). \end{aligned}$$

The case  $n = \sqcup t_i \cdot t_j \sqcup$  is similar.  $\dashv$

LEMMA 13.6. *The following formulas are derived in  $TI_{s+1}$ .*

1.  $\langle n, k \rangle \in \text{Art}(f) \wedge \langle n, q \rangle \in \text{Art}(f) \supset k = q$ .

2.  $\forall n, f^1 \exists! k \{ [\text{Term}(n) \wedge \text{Eval}_0(f) \supset \langle n, k \rangle \in \text{Art}(f)] \wedge [\neg(\text{Term}(n) \wedge \text{Eval}_0(f)) \supset k = 0] \}$ .

PROOF. 1. Proof is by induction on  $n$ . The cases of constants and variables follow from Lemma 13.5.1, 2.

Assume the formula holds for numbers less than  $n$ . We will consider only the case  $n = \sqcup t_i + t_j \sqcup$ , other cases are similar.

Suppose  $\langle n, k \rangle \in \text{Art}(f)$  and  $\langle n, q \rangle \in \text{Art}(f)$ . By Lemma 13.5.4, there exist  $r_1, l_1$  and  $r_2, l_2$  such that

$$\langle i, r_1 \rangle \in \text{Art}(f) \wedge \langle j, l_1 \rangle \in \text{Art}(f) \wedge k = r_1 + l_1$$

and

$$\langle i, r_2 \rangle \in \text{Art}(f) \wedge \langle j, l_2 \rangle \in \text{Art}(f) \wedge q = r_2 + l_2.$$

By the inductive assumption,  $r_1 = r_2$  and  $l_1 = l_2$ , so  $k = q$ .

2. When  $\neg(\text{Term}(n) \wedge \text{Eval}_0(f))$ , it is obvious.

Assume  $\text{Term}(n) \wedge \text{Eval}_0(f)$ . The uniqueness was proven in part 1. We prove the existence by induction on  $n$ .

If  $n = \sqcup 0 \sqcup$ , then  $k = 0$ .

If  $n = \sqcup x_i^0 \sqcup$ , then  $k = \text{Val}_0(f, i)$ .

Assume the existence for numbers less than  $n$ . We will consider only the case  $n = \sqcup t_i + t_j \sqcup$ , other cases are similar. By the inductive assumption, there exist  $r$  and  $l$  such that  $\langle i, r \rangle \in \text{Art}(f)$  and  $\langle j, l \rangle \in \text{Art}(f)$ . Then for  $k = r + l$  we have  $\langle n, k \rangle \in \text{Art}(f)$  by Lemma 13.5.4.  $\dashv$

By the previous lemma we can introduce a functional symbol  $\text{Arval}^0(n, f^1)$  such that

$$TI_{s+1} \vdash \text{Term}(n) \wedge \text{Eval}_0(f^1) \supset \langle n, \text{Arval}(n, f) \rangle \in \text{Art}(f).$$

LEMMA 13.7. *It is derived in  $TI_{s+1}$  that the formula*

$$\text{Eval}_0(f^1)$$

*implies each of the following formulas.*

1.  $Arval(\perp 0 \perp, f) = 0$ .
2.  $Arval(\perp x_i^0 \perp, f) = Val_0(f, i)$ .
3.  $Arval(\perp S(t_j) \perp, f) = S(Arval(j, f))$ .
4.  $Arval(\perp t_i \theta t_j \perp, f) = Arval(i, f) \theta Arval(j, f)$ , where  $\theta$  denotes  $\cdot$  or  $+$ .
5.  $Arval(\perp t \perp, f) = \tilde{t}$ , where  $t$  is any arithmetic term and  $\tilde{t}$  is obtained from  $t$  by replacing each parameter  $x_i^0$  by  $Val_0(f, i)$ .

PROOF. 1. Follows from  $\langle \perp 0 \perp, 0 \rangle \in Art(f)$ , which is in Lemma 13.5.1.

2. Follows from  $\langle \perp x_i^0 \perp, Val_0(f, i) \rangle \in Art(f)$ , which is in Lemma 13.5.2.

3. Denote  $l = Arval(j, f)$ . Then  $\langle j, l \rangle \in Art(f)$  and by Lemma 13.5.3,  $\langle \perp S(t_j) \perp, Sl \rangle \in Art(f)$ , which implies part 3.

4. We prove it for  $+$ , for  $\cdot$  the proof is similar. Denote  $r = Arval(i, f)$  and  $l = Arval(j, f)$ . Then  $\langle i, r \rangle \in Art(f)$ ,  $\langle j, l \rangle \in Art(f)$  and by Lemma 13.5.4,  $\langle \perp t_i + t_j \perp, r + l \rangle \in Art(f)$ , which implies part 4.

5. Proof is by induction on the complexity of  $t$ . The cases of constant 0 and variables follow from parts 1 and 2, respectively.

Assume the formula holds for all proper sub-terms of  $t$ . We will consider only the case  $t = \tau + u$ , other cases are similar. By the inductive assumption:

$$Arval(\perp \tau \perp, f) = \tilde{\tau} \text{ and } Arval(\perp u \perp, f) = \tilde{u}.$$

By part 4,  $Arval(\perp t \perp, f) = \tilde{\tau} + \tilde{u} = \tilde{t}$ . ⊥

We denote  $Form(n)$  the arithmetic formula stating that  $n$  is the Gödel number of a formula of  $TI_s$ . We denote  $Subform(m, n)$  the arithmetic formula stating that  $m$  is the Gödel number of a sub-formula of the formula with Gödel number  $n$ .

Next we introduce a predicate  $Trset(n, b^{s+1})$ , which means that  $b$  is the set of the truth values of all evaluated sub-formulas of  $\varphi_n$ .

$$\begin{aligned} Trset(n, b^{s+1}) \Leftrightarrow & Form(n) \wedge \\ (\forall x^s \in b) \exists m, \bar{f}, \delta^0 [ & x = \langle m, \bar{f}, \delta \rangle \wedge Subform(m, n) \wedge Eval(\bar{f}) \wedge (\delta = 0 \vee \delta = 1)] \\ \wedge \forall m \{ & Subform(m, n) \supset \forall \bar{f} [Eval(\bar{f}) \supset (\langle m, \bar{f}, 0 \rangle \in b \vee \langle m, \bar{f}, 1 \rangle \in b) \\ & \wedge \neg (\langle m, \bar{f}, 0 \rangle \in b \wedge \langle m, \bar{f}, 1 \rangle \in b) \wedge A(m, \bar{f}, b)] \} , \end{aligned}$$

$$\text{where } A(m, \bar{f}, b) = \bigvee_{r=1}^{10} A_r \text{ and } A_1 - A_{10} \text{ are defined as follows.}$$

$$A_1 \Leftrightarrow m = \perp \perp \perp \wedge \langle m, \bar{f}, 0 \rangle \in b.$$

$$A_2 \Leftrightarrow \bigvee_{k=1}^s \exists i, j [m = \perp x_i^k =_k x_j^k \perp \wedge (\langle m, \bar{f}, 1 \rangle \in b \equiv Val_k(f_k, i) =_k Val_k(f_k, j))].$$

$$\begin{aligned} A_3 \Leftrightarrow \bigvee_{k=1}^{s-1} \exists i, j [ & m = \perp x_i^k \in_k x_j^{k+1} \perp \\ & \wedge (\langle m, \bar{f}, 1 \rangle \in b \equiv Val_k(f_k, i) \in_k Val_{k+1}(f_{k+1}, j))] . \end{aligned}$$

$$A_4 \Leftrightarrow \exists i, j \left[ m = \perp t_i =_0 t_j \lrcorner \wedge (\langle m, \bar{f}, 1 \rangle \in b \equiv Arval(i, f_0) =_0 Arval(j, f_0)) \right].$$

$$A_5 \Leftrightarrow \exists i, j \left[ m = \perp t_i \in_0 x_j^1 \lrcorner \wedge (\langle m, \bar{f}, 1 \rangle \in b \equiv Arval(i, f_0) \in_0 Val_1(f_1, j)) \right].$$

Denote the logical connective  $\wedge$  as  $\theta_6$ ,  $\vee$  as  $\theta_7$  and  $\supset$  as  $\theta_8$ . For  $p = 6, 7, 8$ ,

$$A_p \Leftrightarrow \exists i, j \left[ m = \perp \varphi_i \theta_p \varphi_j \lrcorner \wedge (\langle m, \bar{f}, 1 \rangle \in b \equiv (\langle i, \bar{f}, 1 \rangle \in b \theta_p \langle j, \bar{f}, 1 \rangle \in b)) \right].$$

Denote  $Q_9$  the quantifier  $\forall$  and  $Q_{10}$  the quantifier  $\exists$ . For  $p = 9, 10$ ,

$$A_p \Leftrightarrow \bigvee_{k=0}^s \exists i, j \left[ m = \perp Q_p x_i^k \varphi_j \lrcorner \wedge (\langle m, \bar{f}, 1 \rangle \in b \equiv Q_p y^k (\langle j, Sub_k(\bar{f}, i, y), 1 \rangle \in b)) \right].$$

Denote

$$b_{|m} = \{ \langle j, \bar{f}, \delta \rangle \in b \mid Subform(j, m) \}.$$

LEMMA 13.8. *The following formulas are derived in  $TI_{s+1}$ .*

1.  $Subform(m, n) \wedge Trset(n, b^{s+1}) \supset Trset(m, b_{|m})$ .
2.  $Trset(n, b^{s+1}) \wedge Trset(n, c^{s+1}) \supset b = c$ .
3.  $Form(n) \supset \exists! b^{s+1} Trset(n, b)$ .

PROOF. 1. Assume the premises. For any  $j$  with  $Subform(j, m)$  we have:

$$(29) \quad \forall \bar{f} [Eval(\bar{f}) \supset A(j, \bar{f}, b)]$$

and

$$(30) \quad \forall \bar{f}, \delta^0 [\langle j, \bar{f}, \delta \rangle \in b \equiv \langle j, \bar{f}, \delta \rangle \in b_{|m}].$$

To prove  $Trset(m, b_{|m})$ , it is sufficient to show for any  $j$ :

$$(31) \quad Subform(j, m) \supset \forall \bar{f} [Eval(\bar{f}) \supset A(j, \bar{f}, b_{|m})].$$

When  $\varphi_j$  is an atomic formula or  $\perp$ , (31) follows from (29).

Suppose  $j = \perp \exists x_i^k \varphi_p \lrcorner$ . Consider  $\bar{f}$  with  $Eval(\bar{f})$ . By (29),

$$\langle j, \bar{f}, 1 \rangle \in b \equiv \exists y^k (\langle p, Sub_k(\bar{f}, i, y), 1 \rangle \in b)$$

and by (30):

$$\langle j, \bar{f}, 1 \rangle \in b_{|m} \equiv \exists y^k (\langle p, Sub_k(\bar{f}, i, y), 1 \rangle \in b_{|m}),$$

which implies  $A(j, \bar{f}, b_{|m})$ .

In the cases of other quantifiers and logical connectives the proof is similar.

2. Assume the premises. It is sufficient to prove:

$$(32) \quad \forall m [Subform(m, n) \supset (\langle m, \bar{f}, 1 \rangle \in b \equiv \langle m, \bar{f}, 1 \rangle \in c)].$$

If (32) is proven, then for any  $m, \bar{f}$  we have:

$$\langle m, \bar{f}, 0 \rangle \in b \equiv \langle m, \bar{f}, 1 \rangle \notin b \equiv \langle m, \bar{f}, 1 \rangle \notin c \equiv \langle m, \bar{f}, 0 \rangle \in c,$$

and this implies  $b = c$ .

We prove (32) by induction on  $m$ .

Case 1:  $m = \perp \perp \lrcorner$ . Then  $\langle m, \bar{f}, 1 \rangle \notin b$  and  $\langle m, \bar{f}, 1 \rangle \notin c$ .

Case 2:  $m = \perp t_i \in_0 x_j^1 \perp$ . Proof follows from the definition of  $A_5$ . The cases of other atomic formulas are similar.

Assume (32) holds for numbers less than  $m$ .

Case 3:  $m = \perp \exists x_i^k \varphi_j \perp$ . By the inductive assumption:

$$\forall \bar{f} (\langle j, \bar{f}, 1 \rangle \in b \equiv \langle j, \bar{f}, 1 \rangle \in c).$$

So for any  $\bar{f}$ :

$$\begin{aligned} \langle m, \bar{f}, 1 \rangle \in b &\equiv \exists y^k (\langle j, \text{Sub}_k(\bar{f}, i, y), 1 \rangle \in b) \\ &\equiv \exists y^k (\langle j, \text{Sub}_k(\bar{f}, i, y), 1 \rangle \in c) \equiv \langle m, \bar{f}, 1 \rangle \in c. \end{aligned}$$

The cases of other quantifiers and logical connectives are similar.

3. The uniqueness was proven in part 2. We will prove the existence by induction on  $n$ .

Case 1:  $n = \perp \perp \perp$ . We take  $b^{s+1} = \{ \langle n, \bar{f}, 0 \rangle \mid \text{Eval}(\bar{f}) \}$ .

Case 2:  $n = \perp t_i \in_0 x_j^1 \perp$ . We take

$$\begin{aligned} b^{s+1} &= \{ \langle n, \bar{f}, 1 \rangle \mid \text{Eval}(\bar{f}) \wedge \text{Arval}(i, f_0) \in_0 \text{Val}_1(f_1, j) \} \\ &\cup \{ \langle n, \bar{f}, 0 \rangle \mid \text{Eval}(\bar{f}) \wedge \text{Arval}(i, f_0) \notin_0 \text{Val}_1(f_1, j) \}. \end{aligned}$$

Clearly  $\text{Trset}(n, b)$  holds.

Assume the existence for numbers less than  $n$ .

Case 3:  $n = \perp \varphi_i \vee \varphi_j \perp$ . By the inductive assumption there exist sets  $c^1$  and  $d^1$  such that  $\text{Trset}(i, c)$  and  $\text{Trset}(j, d)$ . Denote

$$\varphi(i, j, \bar{f}) \equiv \langle i, \bar{f}, 1 \rangle \in c \vee \langle j, \bar{f}, 1 \rangle \in d.$$

We take

$$\begin{aligned} b^{s+1} &= c \cup d \cup \{ \langle n, \bar{f}, 1 \rangle \mid \text{Eval}(\bar{f}) \wedge \varphi(i, j, \bar{f}) \} \\ &\cup \{ \langle n, \bar{f}, 0 \rangle \mid \text{Eval}(\bar{f}) \wedge \neg \varphi(i, j, \bar{f}) \}. \end{aligned}$$

First we prove:

$$(33) \quad \forall p [\text{Subform}(p, i) \supset \forall \bar{f}, \delta^0 (\langle p, \bar{f}, \delta \rangle \in b \equiv \langle p, \bar{f}, \delta \rangle \in c)].$$

Assume the premises of (33). If  $\neg \text{Subform}(p, j)$ , then the conclusion is obvious. If  $\text{Subform}(p, j)$ , then  $\text{Trset}(p, c|_p)$  and  $\text{Trset}(p, d|_p)$  by part 1,  $c|_p = d|_p$  by part 2 and for any  $\bar{f}, \delta^0$ :

$$\langle p, \bar{f}, \delta \rangle \in b \equiv \langle p, \bar{f}, \delta \rangle \in c|_p \cup d|_p \equiv \langle p, \bar{f}, \delta \rangle \in c|_p \equiv \langle p, \bar{f}, \delta \rangle \in c.$$

(33) is proven. Similarly,

$$(34) \quad \forall p [\text{Subform}(p, j) \supset \forall \bar{f}, \delta^0 (\langle p, \bar{f}, \delta \rangle \in b \equiv \langle p, \bar{f}, \delta \rangle \in d)].$$

To prove  $\text{Trset}(n, b)$ , it is sufficient to show:

$$\begin{aligned} (35) \quad \forall m \{ &\text{Subform}(m, n) \supset \forall \bar{f} [\text{Eval}(\bar{f}) \\ &\supset \neg (\langle m, \bar{f}, 0 \rangle \in b \wedge \langle m, \bar{f}, 1 \rangle \in b) \wedge A(m, \bar{f}, b)] \}. \end{aligned}$$

Assume  $\text{Subform}(m, n)$ .

i)  $m = n$ .

Suppose  $Eval(\bar{f})$ . Then  $\neg(\langle n, \bar{f}, 0 \rangle \in b \wedge \langle n, \bar{f}, 1 \rangle \in b)$  follows from the definition of  $b$ .

Applying (33) to  $p = i$  and (34) to  $p = j$ , we get:

$$\langle n, \bar{f}, 1 \rangle \in b \equiv \varphi(i, j, \bar{f}) \equiv \langle i, \bar{f}, 1 \rangle \in c \vee \langle j, \bar{f}, 1 \rangle \in d \equiv \langle i, \bar{f}, 1 \rangle \in b \vee \langle j, \bar{f}, 1 \rangle \in b,$$

which implies  $A(n, \bar{f}, b)$ .

ii)  $m < n$ .

Then  $Subform(m, i) \vee Subform(m, j)$ . We can assume  $Subform(m, i)$  without loss of generality. Due to  $Trset(i, c)$  we have  $\neg(\langle m, \bar{f}, 0 \rangle \in c \wedge \langle m, \bar{f}, 1 \rangle \in c)$  and by (33),

$$\neg(\langle m, \bar{f}, 0 \rangle \in b \wedge \langle m, \bar{f}, 1 \rangle \in b).$$

$A(m, \bar{f}, b)$  is also derived from  $Trset(i, c)$  by applying (33) to each possible form of the formula  $\varphi_m$ .

This completes the proof for case 3. The cases of other logical connectives and quantifiers are similar.  $\dashv$

LEMMA 13.9. *It is derived in  $TI_{s+1}$  that the formula*

$$Subform(m, n) \wedge Eval(\bar{f}) \wedge Trset(n, b^{s+1})$$

*implies each of the following formulas.*

1.  $\langle m, \bar{f}, 1 \rangle \in b \vee \langle m, \bar{f}, 0 \rangle \in b$ .
2.  $\neg(\langle m, \bar{f}, 1 \rangle \in b \wedge \langle m, \bar{f}, 0 \rangle \in b)$ .
3.  $\langle \perp \perp \perp, \bar{f}, 0 \rangle \in b$ .
4.  $m = \perp x_i^k =_k x_j^k \perp \supset [\langle m, \bar{f}, 1 \rangle \in b \equiv Val_k(f_k, i) =_k Val_k(f_k, j)]$ ,  
 $k = 1, 2, \dots, s$ .
5.  $m = \perp x_i^k \in_k x_j^{k+1} \perp \supset [\langle m, \bar{f}, 1 \rangle \in b \equiv Val_k(f_k, i) \in_k Val_{k+1}(f_{k+1}, j)]$ ,  
 $k = 1, 2, \dots, s-1$ .
6.  $m = \perp t_i =_0 t_j \perp \supset [\langle m, \bar{f}, 1 \rangle \in b \equiv Arval(i, f_0) =_0 Arval(j, f_0)]$ .
7.  $m = \perp t_i \in_0 x_j^1 \perp \supset [\langle m, \bar{f}, 1 \rangle \in b \equiv Arval(i, f_0) \in_0 Val_1(f_1, j)]$ .
8.  $m = \perp \varphi_i \theta \varphi_j \perp \supset [\langle m, \bar{f}, 1 \rangle \in b \equiv (\langle i, \bar{f}, 1 \rangle \in b \theta \langle j, \bar{f}, 1 \rangle \in b)]$ , where  $\theta$  is a logical connective  $\wedge, \vee$  or  $\supset$ .
9.  $m = \perp Q x_i^k \varphi_j \perp \supset [\langle m, \bar{f}, 1 \rangle \in b \equiv Q y^k (\langle j, Sub_k(\bar{f}, i, y), 1 \rangle \in b)]$ ,  
 $k = 0, 1, \dots, s$ , where  $Q$  is a quantifier  $\forall$  or  $\exists$ .

PROOF. Follows from the definition of  $Trset$ .  $\dashv$

We define

$$Truth^{s+1} = \{x^s \mid \exists n, b^{s+1} (Trset(n, b) \wedge x \in b)\}.$$

Thus,  $Truth$  is the set of the truth values of all evaluated formulas of  $TI_s$ .

LEMMA 13.10.  $TI_{s+1} \vdash Subform(m, n) \wedge Trset(n, b^{s+1})$

$$\supset (\langle m, \bar{f}, \delta^0 \rangle \in b \equiv \langle m, \bar{f}, \delta \rangle \in Truth).$$



PROOF. Assume the premises. Then  $Trset(m, b|_m)$ .

$\Rightarrow$  If  $\langle m, \bar{f}, \delta \rangle \in b$ , then  $\langle m, \bar{f}, \delta \rangle \in b|_m$  and  $\langle m, \bar{f}, \delta \rangle \in Truth$  by the definition of the set  $Truth$ .

$\Leftarrow$  Suppose  $\langle m, \bar{f}, \delta \rangle \in Truth$ . Then there is  $c^{s+1}$  with  $Trset(m, c)$  and  $\langle m, \bar{f}, \delta \rangle \in c$ . By Lemma 13.8.2,  $c = b|_m$ , so  $\langle m, \bar{f}, \delta \rangle \in b|_m$  and  $\langle m, \bar{f}, \delta \rangle \in b$ .  $\dashv$

Finally we introduce the truth predicate  $Tr$ :

$$Tr(n, \bar{f}) \Leftarrow \langle n, \bar{f}, 1 \rangle \in Truth.$$

Clearly,  $Tr$  depends on  $s$  but we will omit  $s$  for brevity.

LEMMA 13.11. *It is derived in  $TI_{s+1}$  that the formula*

$$Eval(\bar{f})$$

*implies each of the following formulas.*

1.  $Form(n) \supset (\neg Tr(n, \bar{f}) \equiv \langle n, \bar{f}, 0 \rangle \in Truth)$ .
2.  $Tr(\perp \perp \perp, \bar{f}) \equiv \perp$ .
3.  $Tr(\perp x_i^k =_k x_j^k \perp, \bar{f}) \equiv Val_k(f_k, i) =_k Val_k(f_k, j), k = 1, 2, \dots, s$ .
4.  $Tr(\perp x_i^k \in_k x_j^{k+1} \perp, \bar{f}) \equiv Val_k(f_k, i) \in_k Val_{k+1}(f_{k+1}, j), k = 1, 2, \dots, s-1$ .
5.  $Tr(\perp t_i =_0 t_j \perp, \bar{f}) \equiv Arval(i, f_0) =_0 Arval(j, f_0)$ .
6.  $Tr(\perp t_i \in_0 x_j^1 \perp, \bar{f}) \equiv Arval(i, f_0) \in_0 Val_1(f_1, j)$ .
7.  $Tr(\perp \varphi_i \theta \varphi_j \perp, \bar{f}) \equiv Tr(i, \bar{f}) \theta Tr(j, \bar{f})$ , where  $\theta$  is a logical connective  $\wedge, \vee$  or  $\supset$ .
8.  $Tr(\perp Q x_i^k \varphi_j \perp, \bar{f}) \equiv Q y^k Tr(j, Sub_k(\bar{f}, i, y)), k = 0, 1, \dots, s$ , where  $Q$  is a quantifier  $\forall$  or  $\exists$ .
9.  $Tr(\perp \varphi \perp, \bar{f}) \equiv \tilde{\varphi}$ , where  $\varphi$  is any formula of  $TI_s$  and  $\tilde{\varphi}$  is obtained from  $\varphi$  by replacing each parameter  $x_i^k$  by  $Val_k(f_k, i)$ .
10.  $Tr(\perp \varphi \perp, \bar{f}) \equiv \varphi$  for any closed formula  $\varphi$  of  $TI_s$ .

PROOF. Assume  $Eval(\bar{f})$ . By Lemma 13.8.3, for any  $n$  with  $Form(n)$  there is a set  $b^{s+1}$  such that  $Trset(n, b)$  and by Lemma 13.10:

$$(36) \quad \forall m [Subform(m, n) \supset \forall \bar{f} (Tr(m, \bar{f}) \equiv \langle m, \bar{f}, 1 \rangle \in b)] .$$

1. By (36) for any  $n$  with  $Form(n)$ :

$$\neg Tr(n, \bar{f}) \equiv \langle n, \bar{f}, 1 \rangle \notin b \equiv \langle n, \bar{f}, 0 \rangle \in b \equiv \langle n, \bar{f}, 0 \rangle \in Truth$$

by Lemma 13.9.2, 1 and Lemma 13.10.

2. Denote  $n = \perp \perp \perp$ .

By (36) and Lemma 13.9.3, 2,  $Tr(n, \bar{f}) \equiv \langle n, \bar{f}, 1 \rangle \in b \equiv \perp$ .

Parts 3-5 are similar to part 6, which we consider next.

6. Denote  $n = \perp t_i \in_0 x_j^1 \perp$ . By (36) and Lemma 13.9.7,

$$Tr(n, \bar{f}) \equiv \langle n, \bar{f}, 1 \rangle \in b \equiv Arval(i, f_0) \in_0 Val_1(f_1, j).$$

7. Denote  $n = \perp \varphi_i \vee \varphi_j \perp$ . By (36):

$$\begin{aligned} \forall \bar{f} [Tr(n, \bar{f}) &\equiv \langle n, \bar{f}, 1 \rangle \in b], \\ \forall \bar{f} [Tr(i, \bar{f}) &\equiv \langle i, \bar{f}, 1 \rangle \in b], \\ \forall \bar{f} [Tr(j, \bar{f}) &\equiv \langle j, \bar{f}, 1 \rangle \in b]. \end{aligned}$$

By Lemma 13.9.8,

$$Tr(n, \bar{f}) \equiv \langle n, \bar{f}, 1 \rangle \in b \equiv \langle i, \bar{f}, 1 \rangle \in b \vee \langle j, \bar{f}, 1 \rangle \in b \equiv Tr(i, \bar{f}) \vee Tr(j, \bar{f}).$$

The cases of other logical connectives are similar.

8. Denote  $n = \perp \forall x_i^k \varphi_j \perp$ . By (36):

$$\begin{aligned} \forall \bar{f} [Tr(n, \bar{f}) &\equiv \langle n, \bar{f}, 1 \rangle \in b], \\ \forall \bar{f} [Tr(j, \bar{f}) &\equiv \langle j, \bar{f}, 1 \rangle \in b]. \end{aligned}$$

By Lemma 13.9.9,

$$\begin{aligned} Tr(n, \bar{f}) \equiv \langle n, \bar{f}, 1 \rangle \in b &\equiv \forall y^k (\langle j, Sub_k(\bar{f}, i, y), 1 \rangle \in b) \\ &\equiv \forall y^k Tr(j, Sub_k(\bar{f}, i, y)). \end{aligned}$$

The case of the quantifier  $\exists$  is similar.

9. Proof is by induction on the complexity of  $\varphi$ .

1) If  $\varphi$  is  $\perp$ , then the formula follows from part 2.

2) Suppose  $\varphi$  is  $t \in_0 x_j^1$ . By part 6,

$$Tr(\perp \varphi \perp, \bar{f}) \equiv Arval(\perp t \perp, f_0) \in_0 Val_1(f_1, j) \equiv \tilde{t} \in_0 Val_1(f_1, j) \equiv \tilde{\varphi}$$

by Lemma 13.7.5.

Assume part 9 holds for all proper sub-formulas of  $\varphi$ .

3) Suppose  $\varphi$  is  $\psi \supset \chi$ . By the inductive assumption:

$$Tr(\perp \psi \perp, \bar{f}) \equiv \tilde{\psi} \text{ and } Tr(\perp \chi \perp, \bar{f}) \equiv \tilde{\chi}.$$

By part 7:

$$Tr(\perp \varphi \perp, \bar{f}) \equiv [Tr(\perp \psi \perp, \bar{f}) \supset Tr(\perp \chi \perp, \bar{f})] \equiv (\tilde{\psi} \supset \tilde{\chi}) \equiv \tilde{\varphi}.$$

The cases of other logical connectives are similar.

4) Suppose  $\varphi$  is  $\exists x_j^r \psi(x_j)$ . By part 8:

$$Tr(\perp \varphi \perp, \bar{f}) \equiv \exists y^r Tr(\perp \psi \perp, Sub_r(\bar{f}, j, y)).$$

By the inductive assumption,

$$Tr(\perp \psi \perp, Sub_r(\bar{f}, j, y)) \equiv \tilde{\psi}(y),$$

where  $\tilde{\psi}(y)$  is obtained from  $\psi(x_j)$  by replacing  $x_j$  by  $y$  and each other parameter  $x_i^k$  by  $Val_k(f_k, i)$ . So

$$Tr(\perp \varphi \perp, \bar{f}) \equiv \exists y^r \tilde{\psi}(y) \equiv \tilde{\varphi}.$$

The case of quantifier  $\forall$  is similar.

10. Follows from part 9, since for a closed formula  $\varphi$ ,  $\tilde{\varphi} = \varphi$ . ⊢

THEOREM 13.1.  $TI_{s+1} \vdash Pref_{TI_s}(m) \wedge Eval(\bar{f}) \supset Tr(m, \bar{f})$ .

PROOF. We will describe derivation in  $TI_{s+1}$  informally by induction on the derivation of  $\varphi_m$  in  $TI_s$ .

Suppose  $m$  is the Gödel number of the comprehension axiom:

$$\exists x_j^{n+1} \forall x_i^n (x_i \in x_j \equiv \varphi),$$

where  $srt(\varphi) \leq n+1$  and  $x_j^{n+1}$  is not a parameter of  $\varphi$ .

By Lemma 13.11.10,

$$Tr(\perp\varphi\perp, Sub_n(\bar{f}, i, z)) \equiv \tilde{\varphi},$$

where  $\tilde{\varphi}$  is obtained from  $\varphi$  by replacing parameter  $x_i^n$  by  $z$  and each other parameter  $x_r^k$  by  $Val_k(f_k, r)$ .

If  $k \geq 1$ , then  $Val_k(f_k, r)$  has only a parameter of type  $k$  and  $Val_0(f_1, r)$  has only a parameter of type 1. So  $srt(\tilde{\varphi}) = \max(srt(\varphi), 1) \leq n+1$ . By the comprehension axiom in  $TI_{s+1}$ , there exists a set  $y^{n+1} = \{z^n \mid \tilde{\varphi}\}$ . Then

$$\forall z^n [z \in y \equiv Tr(\perp\varphi\perp, Sub_n(\bar{f}, i, z))],$$

which implies  $Tr(m, \bar{f})$ .

For all other axioms and derivation rules the theorem statement follows from Lemma 13.11.  $\dashv$

**13.2. The strengths of theories  $TI_s$  and  $TI$ .** For an axiomatic theory  $A$  and a fixed Gödel numbering of its expressions denote  $Proof_A(n, m)$  the arithmetic formula stating that a formula with Gödel number  $m$  has a proof with Gödel number  $n$  in the theory  $A$ . Denote  $Prf_A(m) \equiv \exists n Proof_A(n, m)$  and  $Con(A) \equiv \neg Prf_A(\perp\perp)$ . Thus,  $Con(A)$  is the arithmetic formula that states the consistency of  $A$ . Detailed constructions of these formulas can be found, for example, in [10].

For next theorems we fix a trivial evaluation  $\bar{f}$ :  $f_k^k = \{\emptyset^{k-1}\}, k = 1, \dots, s$ , and  $f_0^1 = \{\langle i, 0 \rangle \mid i = i\}$ . Then  $Eval(\bar{f})$ .

**THEOREM 13.2.**  $TI_{s+1} \vdash Con(TI_s), s = 0, 1, 2, \dots$

PROOF. Suppose  $s \geq 1$ . If  $TI_{s+1}$  is inconsistent, then any formula can be derived in it. We assume that it is consistent. We will describe derivation in  $TI_{s+1}$  informally.

Assume  $Prf_{TI_s}(\perp\perp)$ . By Theorem 13.1,  $Tr(\perp\perp, \bar{f})$  that contradicts Lemma 13.11.2. Therefore  $\neg Prf_{TI_s}(\perp\perp)$ .

If  $s = 0$ , then the proof has similar steps, just the definition of the truth predicate  $Tr_0$  does not contain clauses for  $\in_k$ .  $\dashv$

The proofs of the remaining lemmas and theorems in this subsection are similar to the corresponding proofs in [9].

**LEMMA 13.12.**  $TI_2 \vdash Con(I_n) \supset Con(I_{n+1})$ .

PROOF. In Theorem 4.3 in [9] it was proven that  $T_2 \vdash Con(I_n) \supset Con(I_{n+1})$ . The same proof applies to  $TI_2$ .  $\dashv$

**THEOREM 13.3.**  $TI_2 \vdash Con(I_n)$  for  $n = 0, 1, 2, \dots$

PROOF. The proof is by induction on  $n$ .

$n = 0$ :  $TI_0 = I_0 = PA$ , where  $PA$  is the first order arithmetic (Peano arithmetic). By Theorem 13.2,  $TI_1 \vdash Con(I_0)$  and therefore  $TI_2 \vdash Con(I_0)$ .

Assume  $TI_2 \vdash Con(I_n)$ . Then by Lemma 13.12,  $TI_2 \vdash Con(I_{n+1})$ .  $\dashv$

THEOREM 13.4. 1.  $TI_3 \vdash Con(I)$ .  
2.  $TI \vdash Con(I)$ .

PROOF. 1. We will describe derivation in  $TI_3$  informally. Assume  $Prf_I(\perp \perp \perp)$ . Since every proof in  $I$  is finite, for some  $n$  we have:

$$(37) \quad Prf_{I_n}(\perp \perp \perp).$$

By formalizing Theorem 13.3, we get  $Prf_{TI_2}(\perp Con(I_n) \perp)$ . By Theorem 13.1 for  $s = 2$ ,  $Tr(\perp Con(I_n) \perp, \bar{f})$  and by Lemma 13.11.10,  $Con(I_n)$ , since it is a closed formula. This contradicts (37).

2. Follows from part 1.  $\dashv$

#### §14. Comparison of theories $SLP$ and $TI$ .

**14.1. Interpretation of  $SLP_s$  in  $TI_s$ .** Here we formalize our proofs about the model  $\mathcal{B}_s$  in the theory  $TI_s$  for a fixed  $s \geq 1$ . First we introduce some notations in  $TI_s$ .

By induction on  $n$  we define  $(y^n)_i$ .

$(y^0)_i$  is the  $i$ th element of  $y$  when  $y$  is regarded as the numerical code of a finite sequence of natural numbers, it is given by a primitive recursive function.

$$(y^{n+1})_i = \{z^n \mid [\{i+1\}^n, z] \in y\}.$$

Clearly,  $TI_s \vdash (\langle x_1^n, \dots, x_k^n \rangle)_{i-1} = x_i$ , where  $i = 1, 2, \dots, k$ .

All the aforementioned sets can be constructed in  $TI_s$  using its limited comprehension axiom.

For types  $n_1, \dots, n_k$  and  $m = \max\{n_1, \dots, n_k\}$  we define:

$$y_1^{n_1+1} \times \dots \times y_k^{n_k+1} = \left\{ \langle x_1^{n_1}, \dots, x_k^{n_k} \rangle \mid \bigwedge_{i=1}^k (x_i \in y_i) \right\}.$$

Next we introduce a formula  $Seq(x^{n+1}, k)$  of  $TI_s$ , which means that  $x$  is a sequence of length  $k$ .  $Seq(x^0, k)$  is an arithmetic formula stating that the natural number  $x$  is a code for a sequence of natural numbers of length  $k$ .

$$Seq(x^{n+1}, k) \Leftarrow$$

$$\forall u^n (u \in x \equiv u = [\{0\}^n, \{k\}^n] \vee \exists i, z^n (1 \leq i \leq k \wedge u = [\{i\}^n, z])).$$

Clearly,

$$TI_s \vdash \langle x_1^{n_1}, \dots, x_k^{n_k} \rangle = \langle y_1^{n_1}, \dots, y_k^{n_k} \rangle \supset \bigwedge_{i=1}^k (x_i = y_i);$$

$$TI_s \vdash y^m = \langle x_1^{n_1}, \dots, x_k^{n_k} \rangle \supset Seq(y, \tilde{k}),$$

where  $m = \max\{n_1, \dots, n_k\}$ .

The relation  $\leq$  on finite sequences is defined by:

$$x^n \leq y^n \Leftrightarrow \exists k, l [Seq(x, k) \wedge Seq(y, l) \wedge l \leq k \wedge (\forall i < l) ((x)_i = (y)_i)].$$

For fixed  $i$  and  $n \leq m$  the following formula states: "if  $y^m$  is a sequence, then  $x^n$  is its  $i$ th element, and  $\emptyset$  otherwise".

$$\begin{aligned} elt_{m,n}(x^n, y^m, i) \Leftrightarrow \exists k [Seq(y, k) \wedge i < k \wedge (y)_i = \{x\}^{m-n}] \\ \vee [\neg \exists k, z^n (Seq(y, k) \wedge i < k \wedge (y)_i = \{z\}^{m-n}) \wedge x = \emptyset]. \end{aligned}$$

Clearly,  $TI_s \vdash \forall i, y^m \exists! x^n elt_{m,n}(x, y, i)$ .

So we can introduce a functional symbol  $\langle y^m \rangle_{n,i}$  in  $TI_s$  for any  $n \leq m \leq s$ . It is easy to prove that:

$$TI_s \vdash y^m = \langle x_1^{n_1}, \dots, x_k^{n_k} \rangle \supset \langle y \rangle_{n_i, i-1} = x_i.$$

Similarly we introduce a functional symbol  $lth(x)$  such that

$$TI_s \vdash Seq(x^n, k) \supset lth(x) = k.$$

Functions and partial functions are defined as sets of ordered pairs as usual. The predicates "S is a path in  $d$ ", "f is monotonic on  $d$ ", "f is complete on  $d$ " are easily expressed in the language  $TI_s$ . Thus, for a fixed partial order  $\leq$  on  $d$ ,

"S is a path in  $d$ "

$$\Leftrightarrow (\forall x, y \in S)(x \leq y \vee y \leq x) \wedge (\forall x \in d)(\forall y \in S)(x \leq y \vee y \leq x \supset x \in S);$$

"f is monotonic on  $d$ "

$$\Leftrightarrow (\forall x, y \in d)(x \leq y \supset \forall n, z ((\langle y, n \rangle, z) \in f \supset (\langle x, n \rangle, z) \in f));$$

"f is complete on  $d$ "

$$\Leftrightarrow \forall S ["S \text{ is a path in } d" \supset \forall n (\exists x \in S) \exists y ((\langle x, n \rangle, y) \in f)].$$

Here the variables have appropriate types depending on the type of  $d$ .

Next we formalize the Definition 4.1 of the Beth model  $\mathcal{B}_s$  in  $TI_s$ . Most of it is straightforward. We will elaborate only on non-trivial parts. The definition of  $\nu_k(\xi)$  is formalized as follows:

$$\nu_k(\xi) = \{ \langle \langle x, n \rangle, y \rangle \mid n < lh(x) \wedge \langle \langle n, (\langle x \rangle_{k-1, k-1})_n \rangle, y \rangle \in \xi \}.$$

For each  $k = 0, 1, \dots, s-1$ , the sets  $a_k, d_k, b_k, c_k$  and  $l_k$  have type  $k+1$ . The set  $M = d_{s-1}$  has type  $s$ . In theory  $TI_s$  we use predicates (not sets) for  $a_s, b_s, c_s$  and  $l_s$ . For example, for  $a_s$  the corresponding predicate is

$$\begin{aligned} A_s(f^s) \Leftrightarrow f : (d_{s-1} \times \omega) \dashrightarrow a_{s-1} \\ \wedge "f \text{ is monotonic on } d_{s-1}" \wedge "f \text{ is complete on } d_{s-1}" \end{aligned}$$

and it can be expressed by a formula of  $TI_s$ . For universality we will use the notation  $f^s \in a_s$  for the predicate  $A_s(f^s)$ , as well as the notations  $f^s \in b_s$ ,  $f^s \in c_s$  and  $f^s \in l_s$  standing for corresponding predicates.

For  $x, y \in d_k$  we define:

$$x \preceq_k y \Leftrightarrow \bigwedge_{i=0}^k (\langle x \rangle_{i,i} \leq \langle y \rangle_{i,i}).$$

It is easy to see that for any  $y^k \in d_k$  with  $lh(y) = m$  ( $k = 0, 1, \dots, s-1$ ), for any  $n \leq k$  and  $j < m$ :

$$\langle y \rangle_{n,n} \in a_n^{(m)} \text{ and } (\langle y \rangle_{n,n})_j \in a_n.$$

For any  $y^k \in d_k$  and  $n = 1, 2, \dots, k+1$  we denote

$$segment_n(y) = \langle \langle y \rangle_{0,0}, \langle y \rangle_{1,1}, \dots, \langle y \rangle_{n-1,n-1} \rangle.$$

Clearly the type of  $segment_n(y)$  is  $n-1$ . As in the informal definition, we will use the notation  $\bar{y}(n)$  for  $segment_n(y)$ ; it denotes the initial segment of  $y$  of length  $n$ . Here  $\bar{y}(n)$  is introduced only for  $y \in d_k$  and not for arbitrary  $y$ .

Suppose  $\varphi$  is an evaluated formula of  $SLP_s$ . Denote  $int_s(\varphi)$  the formula of  $TI_s$  that expresses the relation  $\mathcal{B}_s \Vdash \varphi$ . For any  $\varphi$ ,  $int_s(\varphi)$  can be formally defined following the Definitions 2.2 and 4.1.

**THEOREM 14.1.**  $SLP_s \vdash \varphi \Rightarrow TI_s \vdash int_s(\bar{\varphi})$ .

**PROOF.** The proofs in Sections 4-9 are easily formalized using the aforementioned notations. That implies the theorem.  $\dashv$

**THEOREM 14.2.** 1. *The theories  $SLP_s$  and  $TI_s$  are equiconsistent.*  
2. *The theories  $SLP$  and  $TI$  are equiconsistent.*

**PROOF.** Part 1 follows from Theorem 12.1.1 and Theorem 14.1. Part 2 follows from Theorem 12.1.2 and part 1, since any derivation in  $SLP$  is finite, therefore it is a derivation in  $SLP_s$  for some  $s$ .  $\dashv$

**14.2. Constructing a forcing predicate for  $SLP_s$  in  $TI_{s+1}$ .** Fix  $s \geq 1$ . In  $TI_{s+1}$  we will construct a formula  $Fr(\alpha, n, \bar{f})$  expressing the forcing predicate  $\alpha \Vdash \varphi_n(\bar{f})$  in the Beth model  $\mathcal{B}_s$ . Thus,  $Fr(\alpha, n, \bar{f})$  means that the formula of  $SLP_s$  with Gödel number  $n$  under evaluation  $\bar{f}$  is forced at node  $\alpha$ . The construction follows similar steps as the construction of the truth predicate  $Tr$  in the previous section, but with a few extra details. In this subsection details of the construction will be described and its applications to the relative strengths of our theories will be given in next sub-section.

We will use the following precise notations for variables of  $SLP_s$ . For  $n = 1, 2, \dots, s$ ;  $i = 1, 2, \dots$ :

- $x_i$  are variables over natural numbers,
- $F_i^n$  are variables over  $n$ -functionals,
- $A_i^n$  are variables over lawlike  $n$ -functionals,
- $\mathcal{F}_i^n$  are variables over lawless  $n$ -functionals.

We will map variables of  $SLP_s$  to variables of  $TI_s$  as follows:

$$\begin{aligned} x_i &\rightarrow x_i^0; \\ F_i^n &\rightarrow x_{3i}^n; \end{aligned}$$

$$A_i^n \rightarrow x_{3i+1}^n;$$

$$\mathcal{F}_i^n \rightarrow x_{3i+2}^n.$$

The definitions of the domains for  $k$ -functionals, lawlike  $k$ -functionals, lawless  $k$ -functionals, and the sets  $d_k$  and  $c_k$  are easily formalised in  $TI_{s+1}$  as in the previous sub-section. All the sets  $a_k$ ,  $b_k$ ,  $l_k$ ,  $d_k$ , and  $c_k$  have type  $k+1$ , including the case  $k=s$  (unlike the previous sub-section, where  $a_s$ ,  $b_s$  and  $l_s$  were predicates). The domain  $M = d_{s-1}$  has type  $s$ .

We will fix Gödel numbering of all expressions of the language  $SLP_s$ . For any expression  $g$  we denote  $\ulcorner g \urcorner$  its Gödel number in this numbering.

The functional symbol  $Val_k(f, i)$  that produces the  $i$ th element of  $f$  was introduced in section 13. Next we introduce a predicate  $Evl_k(f^k)$ ,  $k = 1, \dots, s$ , with the meaning: " $f$  is an evaluation of all variables for  $k$ -functionals".

$$\begin{aligned} Evl_k(f^k) \Leftarrow & (\forall x^{k-1} \in f) \exists i, z^{k-1} (x = \langle i, z \rangle) \\ & \wedge \forall i [Val_k(f, 3i) \in a_k \wedge Val_k(f, 3i+1) \in b_k \wedge Val_k(f, 3i+2) \in l_k]. \end{aligned}$$

For type 0 the predicate  $Evl_0(f^1)$  is the same as  $Eval_0(f^1)$  defined in section 13.

As before we denote  $\bar{f}$  the list of variables  $f_0^1, f_1^1, f_2^2, \dots, f_s^s$  and

$$Evl(\bar{f}) \Leftarrow \bigwedge_{k=0}^s Evl_k(f_k).$$

The functional symbol  $Subst_k(f^k, i, y^k)$  and notation  $Sub_k(\bar{f}, i, y^k)$  were defined in section 13 ( $k = 0, 1, \dots, s$ ).

LEMMA 14.1. *Suppose  $k = 1, 2, \dots, s$ . It is derived in  $TI_{s+1}$  that the formula  $Evl(\bar{f})$  implies each of the following formulas:*

1.  $Evl(Sub_0(\bar{f}, i, u^0));$
2.  $u^k \in a_k \supset Evl(Sub_k(\bar{f}, 3i, u));$
3.  $u^k \in b_k \supset Evl(Sub_k(\bar{f}, 3i+1, u));$
4.  $u^k \in l_k \supset Evl(Sub_k(\bar{f}, 3i+2, u)).$

PROOF. Follows from the definitions of  $Evl$  and  $Sub_k$ . ◻

We denote  $Fnl_k(n)$  the arithmetic formula stating that  $n$  is the Gödel number of a  $k$ -functional ( $k = 1, 2, \dots, s$ ). We denote  $Fnl_0(n)$  the arithmetic formula stating that  $n$  is the Gödel number of a term. We denote  $Z_n$  the term or  $k$ -functional of  $SLP_s$  with Gödel number  $n$  and

$$Subfnl_k(m, n) \Leftarrow Fnl_k(m) \wedge \text{"} Z_m \text{ is a part of } Z_n \text{"}.$$

Thus, for  $k = 1, 2, \dots, s$ ,  $Subfnl_k(m, n)$  means that the  $k$ -functional  $Z_m$  is a sub-functional of  $Z_n$ , and  $Subfnl_0(m, n)$  means that the term  $Z_m$  is a sub-term of  $Z_n$ .

Next we define a predicate  $Fnterm(\alpha, \bar{f}, n, a^{s+1})$ , which means that  $a$  is the set of the values at node  $\alpha$  of all sub-terms and sub-functionals of  $Z_n$  under

evaluation  $\bar{f}$ . The predicate uses the interpretations  $\hat{K}^k$  and  $\hat{N}^k$  from Definition 4.1, which are easily formalised.

$$\begin{aligned} Fnterm(\alpha^{s-1}, n, \bar{f}, a^{s+1}) \Leftrightarrow \alpha \in M \wedge \left( \bigvee_{k=0}^s Fnl_k(n) \right) \wedge Eval(\bar{f}) \\ \wedge (\forall x^s \in a) \exists m \left[ \bigvee_{k=0}^s (Subfnl_k(m, n) \wedge \exists u^k (x = \langle m, u \rangle)) \right] \\ \wedge \forall m \bigwedge_{k=0}^s [Subfnl_k(m, n) \supset C_k(m, a)], \text{ where} \end{aligned}$$

$$C_s(m, a) \Leftrightarrow \bigvee_{r=1}^{10} C_{s,r}; \text{ for } k = 0, 1, \dots, s-1, C_k(m, a) \Leftrightarrow \bigvee_{r=1}^{11} C_{k,r}$$

and  $C_{k,1} - C_{k,11}$  determine the value of  $Z_m$  as follows.

$$C_{k,1} \Leftrightarrow m = \perp 0 \sqcup \wedge \forall y^s (\langle m, y \rangle \in a \equiv y = \{0\}^s).$$

$$C_{k,2} \Leftrightarrow m = \perp K^k \sqcup \wedge \forall y^s \left( \langle m, y \rangle \in a \equiv y = \left\{ \hat{K}^k \right\}^{s-k} \right).$$

$$C_{k,3} \Leftrightarrow \exists i [m = \perp x_i \sqcup \wedge \forall y^s (\langle m, y \rangle \in a \equiv y = \{Val_0(f_0, i)\}^s)].$$

$$C_{k,4} \Leftrightarrow \exists i [m = \perp F_i^k \sqcup \wedge \forall y^s \left( \langle m, y \rangle \in a \equiv y = \{Val_k(f_k, 3i)\}^{s-k} \right)].$$

$$C_{k,5} \Leftrightarrow \exists i [m = \perp A_i^k \sqcup \wedge \forall y^s \left( \langle m, y \rangle \in a \equiv y = \{Val_k(f_k, 3i+1)\}^{s-k} \right)].$$

$$C_{k,6} \Leftrightarrow \exists i [m = \perp \mathcal{F}_i^k \sqcup \wedge \forall y^s \left( \langle m, y \rangle \in a \equiv y = \{Val_k(f_k, 3i+2)\}^{s-k} \right)].$$

$$C_{k,7} \Leftrightarrow \exists j [m = \perp S(Z_j) \sqcup \wedge \forall y^s (\langle m, y \rangle \in a \equiv \exists u^0 (\langle j, u \rangle \in a \wedge y = \{S(u)\}^s))].$$

Denote  $+$  as  $\theta_8$  and  $\cdot$  as  $\theta_9$ . For  $p = 8, 9$ ,

$$\begin{aligned} C_{k,p} \Leftrightarrow \exists j, i [m = \perp Z_j \theta_p Z_i \sqcup \\ \wedge \forall y^s (\langle m, y \rangle \in a \equiv \exists u^0, v^0 (\langle j, u \rangle \in a \wedge \langle i, v \rangle \in a \wedge y = \{u \theta_p v\}^s))]. \end{aligned}$$

$$\begin{aligned} C_{k,10} \Leftrightarrow \exists j \{ m = \perp N^k(Z_j) \sqcup \\ \wedge \forall y^s \left[ \langle m, y \rangle \in a \equiv \exists u^k \left( \langle j, u \rangle \in a \wedge y = \left\{ \hat{N}^k(u) \right\}^{s-k} \right) \right] \}. \end{aligned}$$

$$\begin{aligned} C_{k,11} \Leftrightarrow \exists j, i \{ m = \perp Ap^{k+1}(Z_j, Z_i) \sqcup \wedge \forall y^s [\langle m, y \rangle \in a \\ \equiv \exists w^{k+1}, v^0, u^k (\langle j, w \rangle \in a \wedge \langle i, v \rangle \in a \\ \wedge \langle \langle \bar{\alpha}(k+1), v \rangle, u \rangle \in w \wedge y = \{u\}^{s-k})] \}. \end{aligned}$$



Denote

$$a \parallel_m = \left\{ \langle j, y^s \rangle \in a \mid \bigvee_{p=0}^s \text{Subfnl}_p(j, m) \right\}.$$

LEMMA 14.2. *The following formulas are derived in  $TI_{s+1}$ .*

1.  $\text{Subfnl}_k(m, n) \wedge \text{Fnterm}(\alpha, n, \bar{f}, a^{s+1}) \supset \text{Fnterm}(\alpha, m, \bar{f}, a \parallel_m)$ ,  
 $k = 0, 1, \dots, s$ .
2.  $\text{Fnterm}(\alpha, n, \bar{f}, a^{s+1}) \wedge \text{Fnterm}(\alpha, n, \bar{f}, b^{s+1}) \supset a = b$ .
3.  $\text{Fnl}_k(n) \wedge \alpha \in M \wedge \text{Evl}(\bar{f}) \supset \exists! a^{s+1} \text{Fnterm}(\alpha, n, \bar{f}, a)$ ,  $k = 0, 1, \dots, s$ .

PROOF. 1. Assume the premises. For any  $i$  with  $\text{Subfnl}_p(i, m)$ ,  $p = 0, 1, \dots, s$ , we have:

$$(38) \quad C_p(i, a) \text{ and } \forall y^s (\langle i, y \rangle \in a \equiv \langle i, y \rangle \in a \parallel_m).$$

To prove  $\text{Fnterm}(\alpha, m, \bar{f}, a \parallel_m)$ , it is sufficient to show for any  $j$ :

$$(39) \quad \text{Subfnl}_p(j, m) \supset C_p(j, a \parallel_m).$$

When  $Z_j$  is a constant or a variable, (39) follows from  $C_p(j, a)$ .

Suppose  $j = \perp A p^{p+1}(Z_q, Z_r) \perp$ . Since  $C_p(j, a)$ , we have for any  $y^s$ :

$$\begin{aligned} & \langle j, y \rangle \in a \\ & \equiv \exists w^{p+1}, v^0, u^p [\langle q, w \rangle \in a \wedge \langle r, v \rangle \in a \wedge \langle \bar{\alpha}(p+1), v \rangle, u \rangle \in w \wedge y = \{u\}^{s-p}] \end{aligned}$$

and by (38):

$$\begin{aligned} \langle j, y \rangle \in a \parallel_m & \equiv \exists w^{p+1}, v^0, u^p [\langle q, w \rangle \in a \parallel_m \\ & \wedge \langle r, v \rangle \in a \parallel_m \wedge \langle \bar{\alpha}(p+1), v \rangle, u \rangle \in w \wedge y = \{u\}^{s-p}], \end{aligned}$$

which implies  $C_p(j, a \parallel_m)$ .

For other forms of  $Z_j$  the proof is similar.

2. Assume  $\text{Fnterm}(\alpha, n, \bar{f}, a^{s+1})$  and  $\text{Fnterm}(\alpha, n, \bar{f}, b^{s+1})$ . Then  $\alpha \in M$ ,  $\text{Evl}(\bar{f})$  and  $\text{Fnl}_q(n)$  for some  $q = 0, 1, \dots, s$ . It is sufficient to prove:

$$(40) \quad \forall m \left[ \bigvee_{k=0}^s \text{Subfnl}_k(m, n) \supset \forall y^s (\langle m, y \rangle \in a \equiv \langle m, y \rangle \in b) \right].$$

We will prove it by induction on  $m$ . The cases of constants and variables are obvious.

Assume (40) holds for numbers less than  $m$ . We will consider the only non-trivial case  $m = \perp A p^{k+1}(Z_j, Z_i) \perp$ . By the inductive assumption,

$$\forall y^s (\langle j, y \rangle \in a \equiv \langle j, y \rangle \in b) \text{ and } \forall y^s (\langle i, y \rangle \in a \equiv \langle i, y \rangle \in b).$$

Therefore for any  $y^s$ ,

$$\begin{aligned} & \langle m, y \rangle \in a \\ & \equiv \exists w^{k+1}, v^0, u^k (\langle j, w \rangle \in a \wedge \langle i, v \rangle \in a \wedge \langle \bar{\alpha}(k+1), v \rangle, u \rangle \in w \wedge y = \{u\}^{s-k}) \\ & \equiv \exists w^{k+1}, v^0, u^k (\langle j, w \rangle \in b \wedge \langle i, v \rangle \in b \wedge \langle \bar{\alpha}(k+1), v \rangle, u \rangle \in w \wedge y = \{u\}^{s-k}) \\ & \equiv \langle m, y \rangle \in b. \end{aligned}$$

3. The uniqueness was proven in part 2. We prove the existence by induction on  $n$ .

- 1) If  $n = \perp 0 \perp$ , we take  $a^{s+1} = \{\langle n, 0 \rangle\}$ .
- 2) If  $n = \perp K^k \perp$ , we take  $a^{s+1} = \{\langle n, \hat{K}^k \rangle\}$ .
- 3) If  $n = \perp x_i \perp$ , we take  $a^{s+1} = \{\langle n, Val_0(f_0, i) \rangle\}$ .
- 4) If  $n = \perp F_i^k \perp$ , we take  $a^{s+1} = \{\langle n, Val_k(f_k, 3i) \rangle\}$ .
- 5) If  $n = \perp A_i^k \perp$ , we take  $a^{s+1} = \{\langle n, Val_k(f_k, 3i + 1) \rangle\}$ .
- 6) If  $n = \perp \mathcal{F}_i^k \perp$ , we take  $a^{s+1} = \{\langle n, Val_k(f_k, 3i + 2) \rangle\}$ .

In each case  $Fnterm(\alpha, n, \bar{f}, a)$  is obvious.

Assume the existence for numbers less than  $n$ .

- 7)  $n = \perp Ap^{k+1}(Z_j, Z_i) \perp$ .

By the inductive assumption, there exist sets  $c^{s+1}$  and  $d^{s+1}$  such that

$$Fnterm(\alpha, j, \bar{f}, c) \wedge Fnterm(\alpha, i, \bar{f}, d).$$

We take  $a^{s+1} =$

$$= c \cup d \cup \{\langle n, u^k \rangle \mid \exists w^{k+1}, v^0 (\langle j, w \rangle \in c \wedge \langle i, v \rangle \in d \wedge \langle \bar{\alpha}(k+1), v \rangle, u \rangle \in w)\}$$

and prove  $Fnterm(\alpha, n, \bar{f}, a)$ .

First we prove:

$$(41) \quad \forall p [Subfnl_q(p, j) \supset \forall u^q (\langle p, u \rangle \in a \equiv \langle p, u \rangle \in c)], q = 0, 1, \dots, s.$$

Assume  $Subfnl_q(p, j)$ . If  $\neg Subfnl_q(p, i)$ , then the conclusion is obvious. If  $Subfnl_q(p, i)$ , then  $Fnterm(\alpha, p, \bar{f}, c \parallel_p)$  and  $Fnterm(\alpha, p, \bar{f}, d \parallel_p)$  by part 1,  $c \parallel_p = d \parallel_p$  by part 2 and for any  $u^q$ :

$$\langle p, u \rangle \in a \equiv \langle p, u \rangle \in c \parallel_p \cup d \parallel_p \equiv \langle p, u \rangle \in c.$$

(41) is proven. Similarly,

$$(42) \quad \forall p [Subfnl_q(p, i) \supset \forall u^q (\langle p, u \rangle \in a \equiv \langle p, u \rangle \in d)], q = 0, 1, \dots, s.$$

To prove  $Fnterm(\alpha, n, \bar{f}, a)$ , it is sufficient to show:

$$(43) \quad \forall m \bigwedge_{q=0}^s [Subfnl_q(m, n) \supset C_q(m, a)].$$

Assume  $Subfnl_q(m, n)$ .

i)  $m = n$ .

Then  $q = k$ . Applying (41) to  $p = j$  and (42) to  $p = i$ , we get for any  $y^s$ :

$$\begin{aligned} & \langle n, y \rangle \in a \\ & \equiv \exists w^{k+1}, v^0, u^k (\langle j, w \rangle \in c \wedge \langle i, v \rangle \in d \wedge \langle \bar{\alpha}(k+1), v \rangle, u \rangle \in w \wedge y = \{u\}^{s-k}) \\ & \equiv \exists w^{k+1}, v^0, u^k (\langle j, w \rangle \in a \wedge \langle i, v \rangle \in a \wedge \langle \bar{\alpha}(k+1), v \rangle, u \rangle \in w \wedge y = \{u\}^{s-k}) \end{aligned}$$

which implies  $C_k(n, a)$ .

ii)  $m < n$ .

Then  $Subfnl_q(m, j) \vee Subfnl_q(m, i)$ . We can assume  $Subfnl_q(m, j)$  without loss of generality. Applying (41) to each possible form of  $Z_m$ , we get  $C_q(m, a)$  due to  $Fnterm(\alpha, j, \bar{f}, c)$ .

For example, if  $Z_m = N^q(Z_r)$ , by (41) we have for any  $y^s$ :

$$\begin{aligned} \langle m, y \rangle \in a &\equiv \langle m, y \rangle \in c \equiv \exists u^q \left( \langle r, u \rangle \in c \wedge y = \{\hat{N}^q(u)\}^{s-q} \right) \\ &\equiv \exists u^q \left( \langle r, u \rangle \in a \wedge y = \{\hat{N}^q(u)\}^{s-q} \right), \end{aligned}$$

which implies  $C_q(m, a)$ .

This completes the proof for case 7). The cases of other functional symbols are similar.  $\dashv$

LEMMA 14.3. *Suppose  $k = 0, 1, \dots, s$ . It is derived in  $TI_{s+1}$  that the formula*

$$Subfnl_k(m, n) \wedge Fnterm(\alpha, n, \bar{f}, a^{s+1})$$

*implies each of the following formulas:*

1.  $m = \perp 0 \vdash \forall y^s (\langle m, y \rangle \in a \equiv y = \{0\}^s)$ .
2.  $m = \perp K^k \vdash \forall y^s \left( \langle m, y \rangle \in a \equiv y = \{\hat{K}^k\}^{s-k} \right)$ .
3.  $m = \perp x_i \vdash \forall y^s (\langle m, y \rangle \in a \equiv y = \{Val_0(f_0, i)\}^s)$ .
4.  $m = \perp F_i^k \vdash \forall y^s \left( \langle m, y \rangle \in a \equiv y = \{Val_k(f_k, 3i)\}^{s-k} \right)$ .
5.  $m = \perp A_i^k \vdash \forall y^s \left( \langle m, y \rangle \in a \equiv y = \{Val_k(f_k, 3i+1)\}^{s-k} \right)$ .
6.  $m = \perp \mathcal{F}_i^k \vdash \forall y^s \left( \langle m, y \rangle \in a \equiv y = \{Val_k(f_k, 3i+2)\}^{s-k} \right)$ .
7.  $m = \perp S(Z_j) \vdash \forall y^s (\langle m, y \rangle \in a \equiv \exists u^0 (\langle j, u \rangle \in a \wedge y = \{S(u)\}^s))$ .
8.  $m = \perp Z_j \theta Z_i \vdash \forall y^s (\langle m, y \rangle \in a$   
 $\equiv \exists u^0, v^0 (\langle j, u \rangle \in a \wedge \langle i, v \rangle \in a \wedge y = \{u\theta v\}^s)),$  where  $\theta$  denotes  $\cdot$  or  $+$ .
9.  $m = \perp N^k(Z_j) \vdash \forall y^s \left[ \langle m, y \rangle \in a \equiv \exists u^k \left( \langle j, u \rangle \in a \wedge y = \{\hat{N}^k(u)\}^{s-k} \right) \right]$ .
10.  $m = \perp Ap^{k+1}(Z_j, Z_i) \vdash \forall y^s [\langle m, y \rangle \in a \equiv \exists w^{k+1}, v^0, u^k (\langle j, w \rangle \in a$   
 $\wedge \langle i, v \rangle \in a \wedge \langle \bar{\alpha}(k+1), v \rangle, u \rangle \in w \wedge y = \{u\}^{s-k})]$ .

PROOF. Follows from the definition of  $Fnterm$ .  $\dashv$

Next we define a functional symbol  $Fnt(\alpha, \bar{f})$ :

$$Fnt^{s+1}(\alpha, \bar{f}) = \{ \langle n, y^s \rangle \mid \exists a^{s+1} (Fnterm(\alpha, n, \bar{f}, a) \wedge \langle n, y \rangle \in a) \}.$$

LEMMA 14.4. *The following formulas are derived in  $TI_{s+1}$ .*

1.  $Subfnl_k(m, n) \wedge Fnterm(\alpha, n, \bar{f}, a^{s+1})$   
 $\vdash \forall y^s [\langle m, y \rangle \in a \equiv \langle m, y \rangle \in Fnt(\alpha, \bar{f})], k = 0, 1, \dots, s.$
2.  $\langle n, y^s \rangle \in Fnt(\alpha, \bar{f}) \vdash \bigvee_{k=0}^s [Fnl_k(n) \wedge \exists u^k (y = \{u\}^{s-k})].$

PROOF. 1. Assume the premises. By Lemma 14.2.1,  $Fnterm(\alpha, m, \bar{f}, a \parallel_m)$ . Fix  $y^s$ .

$\Rightarrow$  If  $\langle m, y \rangle \in a$ , then  $\langle m, y \rangle \in a \parallel_m$  and  $\langle m, y \rangle \in Fnt(\alpha, \bar{f})$  by the definition of  $Fnt$ .

$\Leftarrow$  Suppose  $\langle m, y \rangle \in Fnt(\alpha, \bar{f})$ . Then there is  $b^{s+1}$  with  $Fnterm(\alpha, m, \bar{f}, b)$  and  $\langle m, y \rangle \in b$ . By Lemma 14.2.2,  $b = a \parallel_m$ . So  $\langle m, y \rangle \in a \parallel_m$  and  $\langle m, y \rangle \in a$ .

2. Suppose  $\langle n, y^s \rangle \in Fnt(\alpha, \bar{f})$ . Then for some  $a^{s+1}$ ,  $Fnterm(\alpha, n, \bar{f}, a)$  and  $\langle n, y \rangle \in a$ . The conclusion follows from the definition of  $Fnterm$ .  $\dashv$

LEMMA 14.5. *It is derived in  $TI_{s+1}$  that the formula*

$$\alpha \in M \wedge Evl(\bar{f})$$

*implies each of the following formulas.*

1.  $\langle \perp 0 \perp, y^s \rangle \in Fnt(\alpha, \bar{f}) \equiv y = \{0\}^s$ .
2.  $\langle \perp K^k \perp, y^s \rangle \in Fnt(\alpha, \bar{f}) \equiv y = \{\hat{K}^k\}^{s-k}, k = 1, 2, \dots, s$ .
3.  $\langle \perp x_i \perp, y^s \rangle \in Fnt(\alpha, \bar{f}) \equiv y = \{Val_0(f_0, i)\}^s$ .
4.  $\langle \perp F_i^k \perp, y^s \rangle \in Fnt(\alpha, \bar{f}) \equiv y = \{Val_k(f_k, 3i)\}^{s-k}, k = 1, 2, \dots, s$ .
5.  $\langle \perp A_i^k \perp, y^s \rangle \in Fnt(\alpha, \bar{f}) \equiv y = \{Val_k(f_k, 3i+1)\}^{s-k}, k = 1, 2, \dots, s$ .
6.  $\langle \perp F_i^k \perp, y^s \rangle \in Fnt(\alpha, \bar{f}) \equiv y = \{Val_k(f_k, 3i+2)\}^{s-k}, k = 1, 2, \dots, s$ .
7.  $\langle \perp S(Z_j) \perp, y^s \rangle \in Fnt(\alpha, \bar{f}) \equiv \exists u^0 (\langle j, u \rangle \in Fnt(\alpha, \bar{f}) \wedge y = \{S(u)\}^s)$ .
8.  $\langle \perp Z_j \theta Z_i \perp, y^s \rangle \in Fnt(\alpha, \bar{f}) \equiv \exists u^0, v^0 (\langle j, u \rangle \in Fnt(\alpha, \bar{f}) \wedge \langle i, v \rangle \in Fnt(\alpha, \bar{f}) \wedge y = \{u \theta v\}^s)$ , where  $\theta$  denotes  $\cdot$  or  $+$ .
9.  $\langle \perp N^k(Z_j) \perp, y^s \rangle \in Fnt(\alpha, \bar{f}) \equiv \exists u^k (\langle j, u \rangle \in Fnt(\alpha, \bar{f}) \wedge y = \{\hat{N}^k(u)\}^{s-k}), k = 1, 2, \dots, s$ .
10.  $\langle \perp Ap^{k+1}(Z_j, Z_i) \perp, y^s \rangle \in Fnt(\alpha, \bar{f}) \equiv \exists w^{k+1}, v^0, u^k (\langle j, w \rangle \in Fnt(\alpha, \bar{f}) \wedge \langle i, v \rangle \in Fnt(\alpha, \bar{f}) \wedge \langle \bar{\alpha}(k+1), v \rangle, u \rangle \in w \wedge y = \{u\}^{s-k}), k = 0, 1, \dots, s-1$ .

PROOF. Assume  $\alpha \in M \wedge Evl(\bar{f})$ . By Lemma 14.2.3, for any  $n$  with  $Fnl_k(n)$ ,  $k = 0, 1, \dots, s$ , there is a set  $a^{s+1}$  such that  $Fnterm(\alpha, n, \bar{f}, a)$ , and by Lemma 14.4.1:

(44)

$$\forall m [Subfntl_q(m, n) \supset \forall y^s (\langle m, y \rangle \in a \equiv \langle m, y \rangle \in Fnt(\alpha, \bar{f}))], k = 0, 1, \dots, s.$$

1. Denote  $n = \perp 0 \perp$ . By (44), we have for any  $y^s$ :

$$\langle n, y \rangle \in Fnt(\alpha, \bar{f}) \equiv \langle n, y \rangle \in a \equiv y = \{0\}^s$$

by Lemma 14.3.1.

Part 2 is proven similarly. Parts 3, 5 and 6 are similar to part 4, which we will consider next.

4. Denote  $n = \perp F_i^k \perp$ . By (44), we have for any  $y^s$ :

$$\langle n, y \rangle \in Fnt(\alpha, \bar{f}) \equiv \langle n, y \rangle \in a \equiv y = \{Val_k(f_k, 3i)\}^{s-k}$$

by Lemma 14.3.4.

Parts 7-9 are similar to part 10, which we will consider next.

10. Denote  $n = \perp Ap^{k+1}(Z_j, Z_i) \perp$ . By (44):

$$\begin{aligned} \forall y^s (\langle n, y \rangle \in Fnt(\alpha, \bar{f}) &\equiv \langle n, y \rangle \in a); \\ \forall y^s (\langle j, y \rangle \in Fnt(\alpha, \bar{f}) &\equiv \langle j, y \rangle \in a); \\ \forall y^s (\langle i, y \rangle \in Fnt(\alpha, \bar{f}) &\equiv \langle i, y \rangle \in a). \end{aligned}$$

By Lemma 14.3.10, for any  $y^s$ :

$$\begin{aligned} \langle n, y \rangle \in Fnt(\alpha, \bar{f}) &\equiv \langle n, y \rangle \in a \\ \equiv \exists w^{k+1}, v^0, u^k (\langle j, w \rangle \in a \wedge \langle i, v \rangle \in a \wedge \langle \bar{\alpha}(k+1), v \rangle, u \rangle \in w \wedge y = \{u\}^{s-k}) \\ \equiv \exists w^{k+1}, v^0, u^k (\langle j, w \rangle \in Fnt(\alpha, \bar{f}) \wedge \langle i, v \rangle \in Fnt(\alpha, \bar{f}) \\ \wedge \langle \bar{\alpha}(k+1), v \rangle, u \rangle \in w \wedge y = \{u\}^{s-k}). \end{aligned}$$

⊢

LEMMA 14.6. *The following formulas are derived in  $TI_{s+1}$ .*

1.  $Fnl_k(n) \wedge \langle n, u^k \rangle \in Fnt(\alpha, \bar{f}) \supset u \in a_k$ .
2.  $\langle n, u^k \rangle \in Fnt(\alpha, \bar{f}) \wedge \langle n, v^k \rangle \in Fnt(\alpha, \bar{f}) \supset u = v$ .
3.  $\alpha, \beta \in M \wedge \beta \preceq \alpha \supset Fnt(\alpha, \bar{f}) \subseteq Fnt(\beta, \bar{f})$ .
4.  $\alpha \in M \wedge Fnl_k(n) \wedge Evl(\bar{f}) \wedge "S^{s+1} \text{ is a path in } M \text{ through } \alpha"$   
 $\supset (\exists \beta \in S) \exists u^k [\langle n, u \rangle \in Fnt(\beta, \bar{f})]$ .

PROOF. 1. Proof is by induction on  $n$ . The cases of constants follow from Lemma 14.5.1, 2. The cases of variables follow from  $Evl(\bar{f})$  and Lemma 14.5.3-6.

Assume the formula holds for numbers less than  $n$ . We will consider the only non-trivial case  $n = \perp Ap^{k+1}(Z_j, Z_i) \perp$ . Suppose  $\langle n, u^k \rangle \in Fnt(\alpha, \bar{f})$ . Then by Lemma 14.5.10. there exist  $w^{k+1}, v^0$  such that

$$\langle j, w \rangle \in Fnt(\alpha, \bar{f}) \wedge \langle i, v \rangle \in Fnt(\alpha, \bar{f}) \wedge \langle \bar{\alpha}(k+1), v \rangle, u \rangle \in w.$$

By the inductive assumption,  $w \in a_{k+1}$  and by the definition of  $a_{k+1}$ ,  $u \in a_k$ .

2. Proof is by induction on  $n$ . The cases of constants and variables follow from Lemma 14.5.1-6.

Assume the formula holds for numbers less than  $n$ . We will consider the only non-trivial case  $n = \perp Ap^{k+1}(Z_j, Z_i) \perp$ .

Suppose  $\langle n, u^k \rangle \in Fnt(\alpha, \bar{f})$  and  $\langle n, v^k \rangle \in Fnt(\alpha, \bar{f})$ . By Lemma 14.5.10, there exist  $w_1^{k+1}, r_1^0$  and  $w_2^{k+1}, r_2^0$  such that

$$\begin{aligned} \langle j, w_1 \rangle \in Fnt(\alpha, \bar{f}) \wedge \langle i, r_1 \rangle \in Fnt(\alpha, \bar{f}) \wedge \langle \bar{\alpha}(k+1), r_1 \rangle, u \rangle \in w_1 \\ \wedge \langle j, w_2 \rangle \in Fnt(\alpha, \bar{f}) \wedge \langle i, r_2 \rangle \in Fnt(\alpha, \bar{f}) \wedge \langle \bar{\alpha}(k+1), r_2 \rangle, v \rangle \in w_2. \end{aligned}$$

By the inductive assumption,  $w_1 = w_2$  and  $r_1 = r_2$ , so  $\langle \bar{\alpha}(k+1), r_1 \rangle, u \rangle \in w_1$ . Since  $w_1$  is a function, we get  $u = v$ .

3. Assume the premises. It is sufficient to prove that for any  $n$  with  $Fnl_k(n)$ :

$$(45) \quad \langle n, u^k \rangle \in Fnt(\alpha, \bar{f}) \supset \langle n, u \rangle \in Fnt(\beta, \bar{f}), k = 0, 1, \dots, s.$$

We prove (45) by induction on  $n$ . The cases of constants and variables follow from Lemma 14.5.1-6.

Assume (45) holds for numbers less than  $n$ . We consider the only non-trivial case  $n = \perp Ap^{k+1}(Z_j, Z_i) \perp$ . Suppose  $\langle n, u^k \rangle \in Fnt(\alpha, \bar{f})$ . By Lemma 14.5.10, there exist  $w^{k+1}$  and  $v^0$  such that

$$\langle j, w \rangle \in Fnt(\alpha, \bar{f}) \wedge \langle i, v \rangle \in Fnt(\alpha, \bar{f}) \wedge \langle \bar{\alpha}(k+1), v \rangle, u \rangle \in w.$$

By the inductive assumption,  $\langle j, w \rangle \in Fnt(\beta, \bar{f})$  and  $\langle i, v \rangle \in Fnt(\beta, \bar{f})$ . By part 1,  $w \in a_{k+1}$  and is monotonic. Since  $\beta \preceq \alpha$ , we have:

$$\bar{\beta}(k+1) \preceq_k \bar{\alpha}(k+1) \text{ and } \langle \bar{\beta}(k+1), v \rangle, u \rangle \in w,$$

which implies  $\langle n, u \rangle \in Fnt(\beta, \bar{f})$ .

4. Proof is by induction on  $n$ . The cases of constants and variables follow from Lemma 14.5.1-6 by taking  $\beta = \alpha$ .

Assume the formula holds for numbers less than  $n$ . We will consider the only non-trivial case  $n = \perp Ap^{k+1}(Z_j, Z_i) \perp$ .

Suppose " $S^{s+1}$  is a path in  $M$  through  $\alpha$ ". By the inductive assumption there exist  $\beta_1, \beta_2 \in S$  and  $w^{k+1}, v^0$  such that  $\langle j, w \rangle \in Fnt(\beta_1, \bar{f})$  and  $\langle i, v \rangle \in Fnt(\beta_2, \bar{f})$ . By part 1,  $w \in a_{k+1}$ .

Denote  $S_1 = \{\bar{\gamma}(k+1) \mid \gamma \in S\}$ . Then  $S_1$  is a path in  $d_k$  and by completeness of  $w$  there exist  $c \in S_1$  and  $u^k$  such that  $\langle \langle c, v \rangle, u \rangle \in w$ . There is  $\gamma \in S$  with  $\bar{\gamma}(k+1) = c$ .

Denote  $\beta = \min\{\gamma, \beta_1, \beta_2\}$ . Then

$$\beta \preceq \gamma \text{ and } \bar{\beta}(k+1) \preceq_k \bar{\gamma}(k+1),$$

and by monotonicity of  $w$ ,  $\langle \bar{\beta}(k+1), v \rangle, u \rangle \in w$ .

By part 3,  $\langle j, w \rangle \in Fnt(\beta, \bar{f})$  and  $\langle i, v \rangle \in Fnt(\beta, \bar{f})$ . By Lemma 14.5.10, this implies  $\langle n, u \rangle \in Fnt(\beta, \bar{f})$ .  $\dashv$

We denote  $Frm(n)$  the arithmetic formula stating that  $n$  is the Gödel number of a formula of  $SLP_s$ . We denote  $Subfrm(m, n)$  the arithmetic formula stating that  $m$  is the Gödel number of a sub-formula of the formula with Gödel number  $n$ . Next we introduce a predicate  $Forset(n, b^{s+1})$ , which means that  $b$  is the set of forcing values of all evaluated sub-formulas of  $\varphi_n$  at all nodes in  $M$ .

$$\begin{aligned} Forset(n, b^{s+1}) \Leftarrow & Frm(n) \wedge (\forall x^s \in b)(\exists \alpha \in M) \exists m, \bar{f}, \delta [x = \langle \alpha, m, \bar{f}, \delta \rangle \\ & \wedge Subfrm(m, n) \wedge Eval(\bar{f}) \wedge (\delta = 0 \vee \delta = 1)] \wedge \forall m \{ Subfrm(m, n) \\ & \supset (\forall \alpha \in M) \forall \bar{f} [Eval(\bar{f}) \supset (\langle \alpha, m, \bar{f}, 0 \rangle \in b \vee \langle \alpha, m, \bar{f}, 1 \rangle \in b) \\ & \wedge \neg(\langle \alpha, m, \bar{f}, 0 \rangle \in b \wedge \langle \alpha, m, \bar{f}, 1 \rangle \in b) \wedge D(\alpha, m, \bar{f}, b)] \} , \end{aligned}$$

where  $D(\alpha, m, \bar{f}, b) \Leftarrow \bigvee_{r=1}^{13} D_r$ , and  $D_1 - D_{13}$  are defined as follows.

$$D_1 \Leftarrow m = \perp \perp \perp \wedge \langle \alpha, m, \bar{f}, 0 \rangle \in b.$$

$$\begin{aligned}
D_2 &\Leftrightarrow \bigvee_{k=0}^s \exists i, j \{ m = \perp Z_i =_k Z_j \downarrow \wedge [\langle \alpha, m, \bar{f}, 1 \rangle \in b \\
&\quad \equiv \forall S^{s+1} [ "S \text{ is a path in } M \text{ through } \alpha" \\
&\quad \supset (\exists \beta \in S) \exists u^k (\langle i, u \rangle \in Fnt(\beta, \bar{f}) \wedge \langle j, u \rangle \in Fnt(\beta, \bar{f})) ] ] \} .
\end{aligned}$$

$$\begin{aligned}
D_3 &\Leftrightarrow \exists i, j \{ m = \perp \varphi_i \wedge \varphi_j \downarrow \\
&\quad \wedge [\langle \alpha, m, \bar{f}, 1 \rangle \in b \equiv (\langle \alpha, i, \bar{f}, 1 \rangle \in b \wedge \langle \alpha, j, \bar{f}, 1 \rangle \in b)] \} .
\end{aligned}$$

$$\begin{aligned}
D_4 &\Leftrightarrow \exists i, j \{ m = \perp \varphi_i \supset \varphi_j \downarrow \\
&\quad \wedge [\langle \alpha, m, \bar{f}, 1 \rangle \in b \equiv (\forall \beta \in M) (\beta \preceq \alpha \wedge \langle \beta, i, \bar{f}, 1 \rangle \in b \supset \langle \beta, j, \bar{f}, 1 \rangle \in b)] \} .
\end{aligned}$$

$$\begin{aligned}
D_5 &\Leftrightarrow \exists i, j \{ m = \perp \varphi_i \vee \varphi_j \downarrow \wedge [\langle \alpha, m, \bar{f}, 1 \rangle \in b \\
&\quad \equiv \forall S^{s+1} ( "S \text{ is a path in } M \text{ through } \alpha" \\
&\quad \supset (\exists \beta \in S) (\langle \beta, i, \bar{f}, 1 \rangle \in b \vee \langle \beta, j, \bar{f}, 1 \rangle \in b) ) ] \} .
\end{aligned}$$

$$\begin{aligned}
D_6 &\Leftrightarrow \exists i, j \{ m = \perp \forall x_i \varphi_j \downarrow \\
&\quad \wedge [\langle \alpha, m, \bar{f}, 1 \rangle \in b \equiv \forall u^0 (\langle \alpha, j, Sub_0(\bar{f}, i, u), 1 \rangle \in b)] \} .
\end{aligned}$$

$$\begin{aligned}
D_7 &\Leftrightarrow \bigvee_{k=1}^s \exists i, j \{ m = \perp \forall F_i^k \varphi_j \downarrow \\
&\quad \wedge [\langle \alpha, m, \bar{f}, 1 \rangle \in b \equiv (\forall u^k \in a_k) (\langle \alpha, j, Sub_k(\bar{f}, 3i, u), 1 \rangle \in b)] \} .
\end{aligned}$$

$$\begin{aligned}
D_8 &\Leftrightarrow \bigvee_{k=1}^s \exists i, j \{ m = \perp \forall A_i^k \varphi_j \downarrow \\
&\quad \wedge [\langle \alpha, m, \bar{f}, 1 \rangle \in b \equiv (\forall u^k \in b_k) (\langle \alpha, j, Sub_k(\bar{f}, 3i+1, u), 1 \rangle \in b)] \} .
\end{aligned}$$

$$\begin{aligned}
D_9 &\Leftrightarrow \bigvee_{k=1}^s \exists i, j \{ m = \perp \forall \mathcal{F}_i^k \varphi_j \downarrow \\
&\quad \wedge [\langle \alpha, m, \bar{f}, 1 \rangle \in b \equiv (\forall u^k \in l_k) (\langle \alpha, j, Sub_k(\bar{f}, 3i+2, u), 1 \rangle \in b)] \} .
\end{aligned}$$

$$\begin{aligned}
D_{10} &\Leftrightarrow \exists i, j \{ m = \perp \exists x_i \varphi_j \downarrow \\
&\quad \wedge [\langle \alpha, m, \bar{f}, 1 \rangle \in b \equiv \forall S^{s+1} ( "S \text{ is a path in } M \text{ through } \alpha" \\
&\quad \supset (\exists \beta \in S) \exists u^0 (\langle \beta, j, Sub_0(\bar{f}, i, u), 1 \rangle \in b) ) ] \} .
\end{aligned}$$

$$\begin{aligned}
D_{11} &\Leftrightarrow \bigvee_{k=1}^s \exists i, j \{ m = \perp \exists F_i^k \varphi_j \downarrow \\
&\quad \wedge [\langle \alpha, m, \bar{f}, 1 \rangle \in b \equiv \forall S^{s+1} ( "S \text{ is a path in } M \text{ through } \alpha" \\
&\quad \supset (\exists \beta \in S) (\exists u^k \in a_k) (\langle \beta, j, Sub_k(\bar{f}, 3i, u), 1 \rangle \in b) ) ] \} .
\end{aligned}$$

$$D_{12} \Leftrightarrow \bigvee_{k=1}^s \exists i, j \{ m = \perp \exists A_i^k \varphi_j \perp \\ \wedge [ \langle \alpha, m, \bar{f}, 1 \rangle \in b \equiv \forall S^{s+1} ( "S \text{ is a path in } M \text{ through } \alpha" \\ \supset (\exists \beta \in S)(\exists u^k \in b_k) (\langle \beta, j, \text{Sub}_k(\bar{f}, 3i+1, u), 1 \rangle \in b)) ] \} .$$

$$D_{13} \Leftrightarrow \bigvee_{k=1}^s \exists i, j \{ m = \perp \exists \mathcal{F}_i^k \varphi_j \perp \\ \wedge [ \langle \alpha, m, \bar{f}, 1 \rangle \in b \equiv \forall S^{s+1} ( "S \text{ is a path in } M \text{ through } \alpha" \\ \supset (\exists \beta \in S)(\exists u^k \in l_k) (\langle \beta, j, \text{Sub}_k(\bar{f}, 3i+2, u), 1 \rangle \in b)) ] \} .$$

Denote

$$b \mid_m = \{ \langle \alpha, j, \bar{f}, \delta \rangle \in b \mid \text{Subfrm}(j, m) \} .$$

LEMMA 14.7. *The following formulas are derived in  $TI_{s+1}$ .*

1.  $\text{Subfrm}(m, n) \wedge \text{Forset}(n, b^{s+1}) \supset \text{Forset}(m, b \mid_m)$ .
2.  $\text{Forset}(n, b^{s+1}) \wedge \text{Forset}(n, c^{s+1}) \supset b = c$ .
3.  $\text{Frm}(n) \supset \exists! b^{s+1} \text{Forset}(n, b)$ .

PROOF. 1. Assume the premises. For any  $j$  with  $\text{Subfrm}(j, m)$  we have:

$$(46) \quad (\forall \alpha \in M) \forall \bar{f} [ \text{Evl}(\bar{f} \supset D(\alpha, j, \bar{f}, b)) ]$$

and

$$(47) \quad \forall \beta, \bar{f}, \delta [ \langle \beta, j, \bar{f}, \delta \rangle \in b \equiv \langle \beta, j, \bar{f}, \delta \rangle \in b \mid_m ] .$$

To prove  $\text{Forset}(m, b \mid_m)$ , it is sufficient to show for any  $j$ :

$$(48) \quad \text{Subfrm}(j, m) \supset (\forall \alpha \in M) \forall \bar{f} [ \text{Evl}(\bar{f}) \supset D(\alpha, j, \bar{f}, b \mid_m) ] .$$

When  $\varphi_j$  is an atomic formula or  $\perp$ , (48) follows from (46).

Suppose  $j = \perp \exists \mathcal{F}_r^k \varphi_p \perp$ . Consider  $\alpha \in M$  and  $\bar{f}$  with  $\text{Evl}(\bar{f})$ . By (46),

$$\langle \alpha, j, \bar{f}, 1 \rangle \in b \equiv \forall S^{s+1} [ "S \text{ is a path in } M \text{ through } \alpha" \\ \supset (\exists \beta \in S)(\exists u^k \in l_k) (\langle \beta, p, \text{Sub}_k(\bar{f}, 3r+2, u), 1 \rangle \in b) ]$$

and by (47):

$$\langle \alpha, j, \bar{f}, 1 \rangle \in b \mid_m \equiv \forall S^{s+1} [ "S \text{ is a path in } M \text{ through } \alpha" \\ \supset (\exists \beta \in S)(\exists u^k \in l_k) (\langle \beta, p, \text{Sub}_k(\bar{f}, 3r+2, u), 1 \rangle \in b \mid_m) ] ,$$

which implies  $D(\alpha, j, \bar{f}, b \mid_m)$ .

In the cases of other quantifiers and logical connectives the proof is similar.

2. Assume the premises. It is sufficient to prove:

$$(49) \quad \forall m [ \text{Subfrm}(m, n) \supset (\langle \alpha, m, \bar{f}, 1 \rangle \in b \equiv \langle \alpha, m, \bar{f}, 1 \rangle \in c) ] .$$

If (49) is proven, then for any  $\alpha, m, \bar{f}$  we have:

$$\langle \alpha, m, \bar{f}, 0 \rangle \in b \equiv \langle \alpha, m, \bar{f}, 1 \rangle \notin b \equiv \langle \alpha, m, \bar{f}, 1 \rangle \notin c \equiv \langle \alpha, m, \bar{f}, 0 \rangle \in c,$$

and this implies  $b = c$ .



We prove (49) by induction on  $m$ .

Case 1:  $m = \perp \perp \perp$ . Then  $\langle \alpha, m, \bar{f}, 1 \rangle \notin b$  and  $\langle \alpha, m, \bar{f}, 1 \rangle \notin c$ .

Case 2:  $m = \perp Z_i =_k Z_j \perp$ . Proof follows from the definition of  $D_2$ .

Assume (49) holds for numbers less than  $m$ .

Case 3:  $m = \perp \exists \mathcal{F}_i^k \varphi_j \perp$ . By the inductive assumption:

$$\forall \beta, \bar{f} (\langle \beta, j, \bar{f}, 1 \rangle \in b \equiv \langle \beta, j, \bar{f}, 1 \rangle \in c).$$

So for any  $\alpha, \bar{f}$ :

$$\begin{aligned} \langle \alpha, m, \bar{f}, 1 \rangle \in b &\equiv \forall S^{s+1} [{}^n S \text{ is a path in } M \text{ through } \alpha] \\ &\supset (\exists \beta \in S)(\exists u^k \in l_k)(\langle \beta, j, \text{Sub}_k(\bar{f}, 3i+2, u), 1 \rangle \in b) \\ &\equiv \forall S^{s+1} [{}^n S \text{ is a path in } M \text{ through } \alpha] \\ &\supset (\exists \beta \in S)(\exists u^k \in l_k)(\langle \beta, j, \text{Sub}_k(\bar{f}, 3i+2, u), 1 \rangle \in c) \equiv \langle \alpha, m, \bar{f}, 1 \rangle \in c. \end{aligned}$$

The cases of other quantifiers and logical connectives are similar.

3. The uniqueness was proven in part 2. We will prove the existence by induction on  $n$ .

Case 1:  $n = \perp \perp \perp$ . We take  $b^{s+1} = \{\langle \alpha, n, \bar{f}, 0 \rangle \mid \alpha \in M \wedge \text{Evl}(\bar{f})\}$ .

Case 2:  $n = \perp Z_i =_k Z_j \perp$ . Denote

$$\begin{aligned} \varphi(\alpha, i, j, \bar{f}) &\Leftarrow \forall S^{s+1} [{}^n S \text{ is a path in } M \text{ through } \alpha] \\ &\supset (\exists \beta \in S) \exists u^k (\langle i, u \rangle \in \text{Fnt}(\beta, \bar{f}) \wedge \langle j, u \rangle \in \text{Fnt}(\beta, \bar{f}))]. \end{aligned}$$

We take

$$\begin{aligned} b^{s+1} &= \{\langle \alpha, n, \bar{f}, 1 \rangle \mid \alpha \in M \wedge \text{Evl}(\bar{f}) \wedge \varphi(\alpha, i, j, \bar{f})\} \\ &\cup \{\langle \alpha, n, \bar{f}, 0 \rangle \mid \alpha \in M \wedge \text{Evl}(\bar{f}) \wedge \neg \varphi(\alpha, i, j, \bar{f})\}. \end{aligned}$$

Clearly  $\text{Forset}(n, b)$  holds.

Assume the existence for numbers less than  $n$ .

Case 3:  $n = \perp \varphi_i \vee \varphi_j \perp$ . By the inductive assumption there exist sets  $c^{s+1}$  and  $d^{s+1}$  such that  $\text{Forset}(i, c)$  and  $\text{Forset}(j, d)$ . Denote

$$\begin{aligned} \psi(\alpha, i, j, \bar{f}) &\Leftarrow \forall S^{s+1} [{}^n S \text{ is a path in } M \text{ through } \alpha] \\ &\supset (\exists \beta \in S)(\langle \beta, i, \bar{f}, 1 \rangle \in c \vee \langle \beta, j, \bar{f}, 1 \rangle \in d)]. \end{aligned}$$

We take

$$\begin{aligned} b^{s+1} &= c \cup d \cup \{\langle \alpha, n, \bar{f}, 1 \rangle \mid \alpha \in M \wedge \text{Evl}(\bar{f}) \wedge \psi(\alpha, i, j, \bar{f})\} \\ &\cup \{\langle \alpha, n, \bar{f}, 0 \rangle \mid \alpha \in M \wedge \text{Evl}(\bar{f}) \wedge \neg \psi(\alpha, i, j, \bar{f})\}. \end{aligned}$$

First we prove:

$$(50) \quad \forall p [ \text{Subfrm}(p, i) \supset \forall \beta, \bar{f}, \delta (\langle \beta, p, \bar{f}, \delta \rangle \in b \equiv \langle \beta, p, \bar{f}, \delta \rangle \in c) ].$$

Assume the premises of (50). If  $\neg \text{Subfrm}(p, j)$ , then the conclusion is obvious. If  $\text{Subfrm}(p, j)$ , then  $\text{Forset}(p, c \upharpoonright_p)$  and  $\text{Forset}(p, d \upharpoonright_p)$  by part 1,  $c \upharpoonright_p = d \upharpoonright_p$  by part 2 and for any  $\beta, \bar{f}, \delta$ :

$$\langle \beta, p, \bar{f}, \delta \rangle \in b \equiv \langle \beta, p, \bar{f}, \delta \rangle \in c \upharpoonright_p \cup d \upharpoonright_p \equiv \langle \beta, p, \bar{f}, \delta \rangle \in c.$$

(50) is proven. Similarly,

$$(51) \quad \forall p [Subfrm(p, j) \supset \forall \beta, \bar{f}, \delta (\langle \beta, p, \bar{f}, \delta \rangle \in b \equiv \langle \beta, p, \bar{f}, \delta \rangle \in d)].$$

To prove  $Forset(n, b)$ , it is sufficient to show:

$$(52) \quad \forall m \{ Subfrm(m, n) \supset (\forall \alpha \in M) \forall \bar{f} [Evl(\bar{f}) \\ \supset \neg(\langle \alpha, m, \bar{f}, 0 \rangle \in b \wedge \langle \alpha, m, \bar{f}, 1 \rangle \in b) \wedge D(\alpha, m, \bar{f}, b)] \}.$$

Assume  $Subfrm(m, n)$ .

i)  $m = n$ .

Suppose  $\alpha \in M$  and  $Evl(\bar{f})$ . Then  $\neg(\langle \alpha, n, \bar{f}, 0 \rangle \in b \wedge \langle \alpha, n, \bar{f}, 1 \rangle \in b)$  follows from the definition of  $b$ .

Applying (50) to  $p = i$  and (51) to  $p = j$ , we get:

$$\begin{aligned} \langle \alpha, n, \bar{f}, 1 \rangle \in b &\equiv \psi(\alpha, i, j, \bar{f}) \equiv \forall S^{s+1} ["S \text{ is a path in } M \text{ through } \alpha"] \\ &\supset (\exists \beta \in S)(\langle \beta, i, \bar{f}, 1 \rangle \in c \vee \langle \beta, j, \bar{f}, 1 \rangle \in d) \\ &\equiv \forall S^{s+1} ["S \text{ is a path in } M \text{ through } \alpha"] \\ &\supset (\exists \beta \in S)(\langle \beta, i, \bar{f}, 1 \rangle \in b \vee \langle \beta, j, \bar{f}, 1 \rangle \in b), \end{aligned}$$

which implies  $D(\alpha, n, \bar{f}, b)$ .

ii)  $m < n$ .

Then  $Subfrm(m, i) \vee Subfrm(m, j)$ . We can assume  $Subfrm(m, i)$  without loss of generality. Due to  $Forset(i, c)$  we have  $\neg(\langle \alpha, m, \bar{f}, 0 \rangle \in c \wedge \langle \alpha, m, \bar{f}, 1 \rangle \in c)$  and by (50),

$$\neg(\langle \alpha, m, \bar{f}, 0 \rangle \in b \wedge \langle \alpha, m, \bar{f}, 1 \rangle \in b).$$

$D(\alpha, m, \bar{f}, b)$  is also derived from  $Forset(i, c)$  by applying (50) to each possible form of the formula  $\varphi_m$ .

This completes the proof for case 3. The cases of other logical connectives and quantifiers are similar.  $\dashv$

LEMMA 14.8. *It is derived in  $TI_{s+1}$  that the formula*

$$\alpha \in M \wedge Subfrm(m, n) \wedge Evl(\bar{f}) \wedge Forset(n, b^{s+1})$$

*implies each of the following formulas.*

1.  $\langle \alpha, m, \bar{f}, 1 \rangle \in b \vee \langle \alpha, m, \bar{f}, 0 \rangle \in b$ .
2.  $\neg(\langle \alpha, m, \bar{f}, 1 \rangle \in b \wedge \langle \alpha, m, \bar{f}, 0 \rangle \in b)$ .
3.  $\langle \alpha, \perp \perp \perp, \bar{f}, 0 \rangle \in b$ .
4.  $m = \perp Z_i =_k Z_j \supset \{ \langle \alpha, m, \bar{f}, 1 \rangle \in b \equiv \forall S^{s+1} ["S \text{ is a path in } M \\ \text{through } \alpha"] \supset (\exists \beta \in S) \exists u^k (\langle i, u \rangle \in Fnt(\beta, \bar{f}) \wedge \langle j, u \rangle \in Fnt(\beta, \bar{f})) \} , \\ k = 0, 1, \dots, s.$
5.  $m = \perp \varphi_i \wedge \varphi_j \supset [ \langle \alpha, m, \bar{f}, 1 \rangle \in b \equiv (\langle \alpha, i, \bar{f}, 1 \rangle \in b \wedge \langle \alpha, j, \bar{f}, 1 \rangle \in b) ]$ .
6.  $m = \perp \varphi_i \supset \varphi_j \supset [ \langle \alpha, m, \bar{f}, 1 \rangle \in b \equiv (\forall \beta \in M) (\beta \preceq \alpha \wedge \langle \beta, i, \bar{f}, 1 \rangle \in b \supset \langle \beta, j, \bar{f}, 1 \rangle \in b) ]$ .

7.  $m = \perp \varphi_i \vee \varphi_j \supset [\langle \alpha, m, \bar{f}, 1 \rangle \in b \equiv \forall S^{s+1} ("S \text{ is a path in } M \text{ through } \alpha"$   
 $\supset (\exists \beta \in S) (\langle \beta, i, \bar{f}, 1 \rangle \in b \vee \langle \beta, j, \bar{f}, 1 \rangle \in b))]$ .
8.  $m = \perp \forall x_i \varphi_j \supset [\langle \alpha, m, \bar{f}, 1 \rangle \in b \equiv \forall u^0 (\langle \alpha, j, Sub_0(\bar{f}, i, u), 1 \rangle \in b)]$ .
9.  $m = \perp \forall F_i^k \varphi_j \supset [\langle \alpha, m, \bar{f}, 1 \rangle \in b$   
 $\equiv (\forall u^k \in a_k) (\langle \alpha, j, Sub_k(\bar{f}, 3i, u), 1 \rangle \in b)], k = 1, 2, \dots, s$ .
10.  $m = \perp \forall A_i^k \varphi_j \supset [\langle \alpha, m, \bar{f}, 1 \rangle \in b$   
 $\equiv (\forall u^k \in b_k) (\langle \alpha, j, Sub_k(\bar{f}, 3i + 1, u), 1 \rangle \in b)], k = 1, 2, \dots, s$ .
11.  $m = \perp \forall \mathcal{F}_i^k \varphi_j \supset [\langle \alpha, m, \bar{f}, 1 \rangle \in b$   
 $\equiv (\forall u^k \in l_k) (\langle \alpha, j, Sub_k(\bar{f}, 3i + 2, u), 1 \rangle \in b)], k = 1, 2, \dots, s$ .
12.  $m = \perp \exists x_i \varphi_j \supset \{ \langle \alpha, m, \bar{f}, 1 \rangle \in b \equiv \forall S^{s+1} ["S \text{ is a path in } M \text{ through } \alpha"$   
 $\supset (\exists \beta \in S) \exists u^0 (\langle \beta, j, Sub_0(\bar{f}, i, u), 1 \rangle \in b)] \}$ .
13.  $m = \perp \exists F_i^k \varphi_j \supset \{ \langle \alpha, m, \bar{f}, 1 \rangle \in b \equiv \forall S^{s+1} ["S \text{ is a path in } M \text{ through } \alpha"$   
 $\supset (\exists \beta \in S) (\exists u^k \in a_k) (\langle \beta, j, Sub_k(\bar{f}, 3i, u), 1 \rangle \in b)] \}, k = 1, 2, \dots, s$ .
14.  $m = \perp \exists A_i^k \varphi_j \supset \{ \langle \alpha, m, \bar{f}, 1 \rangle \in b \equiv \forall S^{s+1} ["S \text{ is a path in } M \text{ through } \alpha"$   
 $\supset (\exists \beta \in S) (\exists u^k \in b_k) (\langle \beta, j, Sub_k(\bar{f}, 3i + 1, u), 1 \rangle \in b)] \}, k = 1, 2, \dots, s$ .
15.  $m = \perp \exists \mathcal{F}_i^k \varphi_j \supset \{ \langle \alpha, m, \bar{f}, 1 \rangle \in b \equiv \forall S^{s+1} ["S \text{ is a path in } M \text{ through } \alpha"$   
 $\supset (\exists \beta \in S) (\exists u^k \in l_k) (\langle \beta, j, Sub_k(\bar{f}, 3i + 2, u), 1 \rangle \in b)] \}, k = 1, 2, \dots, s$ .

PROOF. Follows from the definition of *Forset*.  $\dashv$

We define

$$Force^{s+1} = \{x^s \mid \exists n, b^{s+1} (Forset(n, b) \wedge x \in b)\}.$$

Thus, *Force* is the set of the forcing values of all evaluated formulas of  $SLP_s$  at all nodes of the Beth model  $\mathcal{B}_s$ .

LEMMA 14.9.

$$TI_{s+1} \vdash Forset(n, b^{s+1}) \wedge Subfrm(m, n) \\ \supset (\langle \alpha, m, \bar{f}, \delta \rangle \in b \equiv \langle \alpha, m, \bar{f}, \delta \rangle \in Force).$$

PROOF. Assume the premises. Then  $Forset(m, b \upharpoonright_m)$  by Lemma 14.7.1.

$\Rightarrow$  If  $\langle \alpha, m, \bar{f}, \delta \rangle \in b$ , then  $\langle \alpha, m, \bar{f}, \delta \rangle \in b \upharpoonright_m$  and  $\langle \alpha, m, \bar{f}, \delta \rangle \in Force$  by the definition of the set *Force*.

$\Leftarrow$  Suppose  $\langle \alpha, m, \bar{f}, \delta \rangle \in Force$ . Then there is  $c^{s+1}$  with  $Forset(m, c)$  and  $\langle \alpha, m, \bar{f}, \delta \rangle \in c$ . By Lemma 14.7.2,  $c = b \upharpoonright_m$ , so  $\langle \alpha, m, \bar{f}, \delta \rangle \in b \upharpoonright_m \subseteq b$ .  $\dashv$

Finally we introduce the forcing predicate *Fr*:

$$Fr(\alpha, n, \bar{f}) \equiv \langle \alpha, n, \bar{f}, 1 \rangle \in Force.$$

Clearly, *Fr* depends on  $s$  but we will omit  $s$  for brevity.

LEMMA 14.10. *It is derived in  $TI_{s+1}$  that the formula*

$$\alpha \in M \wedge \text{Evl}(\bar{f})$$

*implies each of the following formulas.*

1.  $\text{Frm}(n) \supset (\neg \text{Fr}(\alpha, n, \bar{f}) \equiv \langle \alpha, n, \bar{f}, 0 \rangle \in \text{Force})$ .
2.  $\text{Fr}(\alpha, \perp \perp \perp, \bar{f}) \equiv \perp$ .
3.  $\text{Fr}(\alpha, \perp Z_i =_k Z_j \perp, \bar{f}) \equiv \forall S^{s+1} [{}^n S \text{ is a path in } M \text{ through } \alpha]$   
 $\supset (\exists \beta \in S) \exists u^k (\langle i, u \rangle \in \text{Fnt}(\beta, \bar{f}) \wedge \langle j, u \rangle \in \text{Fnt}(\beta, \bar{f}))], k = 0, 1, \dots, s$ .
4.  $\text{Fr}(\alpha, \perp \varphi_i \wedge \varphi_j \perp, \bar{f}) \equiv \text{Fr}(\alpha, i, \bar{f}) \wedge \text{Fr}(\alpha, j, \bar{f})$ .
5.  $\text{Fr}(\alpha, \perp \varphi_i \supset \varphi_j \perp, \bar{f}) \equiv (\forall \beta \in M) [\beta \preceq \alpha \wedge \text{Fr}(\alpha, i, \bar{f}) \supset \text{Fr}(\alpha, j, \bar{f})]$ .
6.  $\text{Fr}(\alpha, \perp \varphi_i \vee \varphi_j \perp, \bar{f}) \equiv \forall S^{s+1} [{}^n S \text{ is a path in } M \text{ through } \alpha]$   
 $\supset (\exists \beta \in S) (\text{Fr}(\beta, i, \bar{f}) \vee \text{Fr}(\beta, j, \bar{f}))]$ .
7.  $\text{Fr}(\alpha, \perp \forall x_i \varphi_j \perp, \bar{f}) \equiv \forall u^0 \text{Fr}(\alpha, j, \text{Sub}_0(\bar{f}, i, u))$ .
8.  $\text{Fr}(\alpha, \perp \forall F_i^k \varphi_j \perp, \bar{f}) \equiv (\forall u^k \in a_k) \text{Fr}(\alpha, j, \text{Sub}_k(\bar{f}, 3i, u)), k = 1, 2, \dots, s$ .
9.  $\text{Fr}(\alpha, \perp \forall A_i^k \varphi_j \perp, \bar{f}) \equiv (\forall u^k \in b_k) \text{Fr}(\alpha, j, \text{Sub}_k(\bar{f}, 3i + 1, u)), k = 1, 2, \dots, s$ .
10.  $\text{Fr}(\alpha, \perp \forall \mathcal{F}_i^k \varphi_j \perp, \bar{f}) \equiv (\forall u^k \in l_k) \text{Fr}(\alpha, j, \text{Sub}_k(\bar{f}, 3i + 2, u)), k = 1, 2, \dots, s$ .
11.  $\text{Fr}(\alpha, \perp \exists x_i \varphi_j \perp, \bar{f}) \equiv \forall S^{s+1} [{}^n S \text{ is a path in } M \text{ through } \alpha]$   
 $\supset (\exists \beta \in S) \exists u^0 \text{Fr}(\beta, j, \text{Sub}_0(\bar{f}, i, u))]$ .
12.  $\text{Fr}(\alpha, \perp \exists F_i^k \varphi_j \perp, \bar{f}) \equiv \forall S^{s+1} [{}^n S \text{ is a path in } M \text{ through } \alpha]$   
 $\supset (\exists \beta \in S) (\exists u^k \in a_k) \text{Fr}(\beta, j, \text{Sub}_k(\bar{f}, 3i, u))], k = 1, 2, \dots, s$ .
13.  $\text{Fr}(\alpha, \perp \exists A_i^k \varphi_j \perp, \bar{f}) \equiv \forall S^{s+1} [{}^n S \text{ is a path in } M \text{ through } \alpha]$   
 $\supset (\exists \beta \in S) (\exists u^k \in b_k) \text{Fr}(\beta, j, \text{Sub}_k(\bar{f}, 3i + 1, u))], k = 1, 2, \dots, s$ .
14.  $\text{Fr}(\alpha, \perp \exists \mathcal{F}_i^k \varphi_j \perp, \bar{f}) \equiv \forall S^{s+1} [{}^n S \text{ is a path in } M \text{ through } \alpha]$   
 $\supset (\exists \beta \in S) (\exists u^k \in l_k) \text{Fr}(\beta, j, \text{Sub}_k(\bar{f}, 3i + 2, u))], k = 1, 2, \dots, s$ .

PROOF. Assume  $\alpha \in M \wedge \text{Evl}(\bar{f})$ . By Lemma 14.7.3, for any  $n$  with  $\text{Frm}(n)$  there is a set  $b^{s+1}$  such that  $\text{Forset}(n, b)$  and by Lemma 14.9:

$$(53) \quad \forall m [\text{Subfrm}(m, n) \supset \forall \beta, \bar{f} (\text{Fr}(\beta, m, \bar{f}) \equiv \langle \beta, m, \bar{f}, 1 \rangle \in b)]$$

1. By (53) for any  $n$  with  $\text{Frm}(n)$ :

$$\neg \text{Fr}(\alpha, n, \bar{f}) \equiv \langle \alpha, n, \bar{f}, 1 \rangle \notin b \equiv \langle \alpha, n, \bar{f}, 0 \rangle \in b \equiv \langle \alpha, n, \bar{f}, 0 \rangle \in \text{Force}$$

by Lemma 14.8.2, 1 and Lemma 14.9.

2. Denote  $n = \perp \perp \perp$ .

By (53) and Lemma 14.8.3, 2,  $\text{Fr}(\alpha, n, \bar{f}) \equiv \langle \alpha, n, \bar{f}, 1 \rangle \in b \equiv \perp$ .

3. Denote  $n = \perp Z_i =_k Z_j \perp$ . By (53) and Lemma 14.8.4,

$$\begin{aligned} Fr(\alpha, n, \bar{f}) \equiv \langle \alpha, n, \bar{f}, 1 \rangle \in b &\equiv \forall S^{s+1} [{}^n S \text{ is a path in } M \text{ through } \alpha] \\ &\supset (\exists \beta \in S) \exists u^k (\langle i, u \rangle \in Fnt(\beta, \bar{f}) \wedge \langle j, u \rangle \in Fnt(\beta, \bar{f}))]. \end{aligned}$$

Parts 4 and 5 are similar to part 6, which we consider next.

6. Denote  $n = \perp \varphi_i \vee \varphi_j \perp$ . By (53):

$$\begin{aligned} \forall \beta, \bar{f} [Fr(\beta, n, \bar{f}) \equiv \langle \beta, n, \bar{f}, 1 \rangle \in b], \\ \forall \beta, \bar{f} [Fr(\beta, i, \bar{f}) \equiv \langle \beta, i, \bar{f}, 1 \rangle \in b], \\ \forall \beta, \bar{f} [Fr(\beta, j, \bar{f}) \equiv \langle \beta, j, \bar{f}, 1 \rangle \in b]. \end{aligned}$$

By Lemma 14.8.7,

$$\begin{aligned} Fr(\alpha, n, \bar{f}) \equiv \langle \alpha, n, \bar{f}, 1 \rangle \in b &\equiv \forall S^{s+1} [{}^n S \text{ is a path in } M \text{ through } \alpha] \\ &\supset (\exists \beta \in S) (\langle \beta, i, \bar{f}, 1 \rangle \in b \vee \langle \beta, j, \bar{f}, 1 \rangle \in b) \\ &\equiv \forall S^{s+1} [{}^n S \text{ is a path in } M \text{ through } \alpha] \\ &\supset (\exists \beta \in S) (Fr(\beta, i, \bar{f}) \vee Fr(\beta, j, \bar{f})). \end{aligned}$$

Parts 7-13 are similar to part 14, which we consider next.

14. Denote  $n = \perp \exists \mathcal{F}_i^k \varphi_j \perp$ . By (53):

$$\begin{aligned} \forall \beta, \bar{f} [Fr(\beta, n, \bar{f}) \equiv \langle \beta, n, \bar{f}, 1 \rangle \in b], \\ \forall \beta, \bar{f} [Fr(\beta, j, \bar{f}) \equiv \langle \beta, j, \bar{f}, 1 \rangle \in b]. \end{aligned}$$

By Lemma 14.8.15,

$$\begin{aligned} Fr(\alpha, n, \bar{f}) \equiv \langle \alpha, n, \bar{f}, 1 \rangle \in b &\equiv \forall S^{s+1} [{}^n S \text{ is a path in } M \text{ through } \alpha] \\ &\supset (\exists \beta \in S) (\exists u^k \in l_k) (\langle \beta, j, Sub_k(\bar{f}, 3i+2, u), 1 \rangle \in b) \\ &\equiv \forall S^{s+1} [{}^n S \text{ is a path in } M \text{ through } \alpha] \\ &\supset (\exists \beta \in S) (\exists u^k \in l_k) Fr(\beta, j, Sub_k(\bar{f}, 3i+2, u)). \end{aligned}$$

⊣

**THEOREM 14.3.**  $TI_{s+1} \vdash Prf_{SLP_s}(m) \wedge Evl(\bar{f}) \supset Fr(\varepsilon, m, \bar{f})$ , where  $\varepsilon$  was defined in Definition 4.1.

**PROOF.** Informal proofs in sections 4, 6, 7, 8 and 9 are easily formalised in  $TI_{s+1}$ , due to Lemmas 14.10, 14.5 and 14.6. ⊣

### 14.3. The strengths of theories $SLP_s$ and $SLP$ .

**THEOREM 14.4.**  $TI_{s+1} \vdash Con(SLP_s), s = 0, 1, 2, \dots$

**PROOF.** Suppose  $s \geq 1$ . If  $TI_{s+1}$  is inconsistent, then any formula can be derived in it. We assume that it is consistent. We will describe derivation in  $TI_{s+1}$  informally.

For any  $k = 1, 2, \dots, s$  we fix trivial elements in the domains  $a_k, b_k$  and  $l_k$ :

$$\begin{aligned} g_{0k} &= g_{1k} = \hat{K}^k; \\ g_{2k} &= \nu_k(\xi), \text{ where } \xi = \{ \langle \langle n, y^k \rangle, y^k \rangle \mid y \in a_{k-1} \}. \end{aligned}$$

Thus,  $g_{0k} \in a_k, g_{1k} \in b_k$  and  $g_{2k} \in l_k$ . Next we fix a trivial evaluation  $\bar{f}$ :

$$f_0^1 = \{\langle i, 0 \rangle \mid i = i\}; \text{ for } k = 1, 2, \dots, s:$$

$$f_k^k = \{\langle 3i, z^{k-1} \rangle \mid i \in \omega \wedge z \in g_{0k}\} \cup \{\langle 3i + 1, z^{k-1} \rangle \mid i \in \omega \wedge z \in g_{1k}\} \\ \cup \{\langle 3i + 2, z^{k-1} \rangle \mid i \in \omega \wedge z \in g_{2k}\}.$$

Then  $Evl(\bar{f})$ .

Assume  $Prf_{SLP_s}(\perp \perp \perp)$ . By Theorem 14.3,  $Fr(\varepsilon, \perp \perp \perp, \bar{f})$  that contradicts Lemma 14.10.2. Therefore  $\neg Prf_{SLP_s}(\perp \perp \perp)$ .

Now suppose  $s = 0$ . Clearly  $SLP_0$  is the intuitionistic first-order arithmetic  $HA$  and  $TI_1$  is the classical second-order arithmetic, the same as  $T_1$ . The theorem follows from the well-known fact that  $T_1 \vdash Con(HA)$ .  $\dashv$

**THEOREM 14.5.** For  $s = 0, 1, 2, \dots$ ,

1.  $SLP_{s+1} \vdash Con(TI_s)$ ;
2.  $SLP_{s+1} \vdash Con(SLP_s)$ .

**PROOF.** 1. By Theorem 13.2, we have  $TI_{s+1} \vdash Con(TI_s)$ . By Theorem 12.1.1, we have  $SLP_{s+1} \vdash int(Con(TI_s))$ . By Lemma 12.3,  $SLP_{s+1} \vdash Con(TI_s)$ , since  $Con(TI_s) \equiv \forall n \neg Proof_{TI_s}(n, \perp \perp \perp)$  and  $Proof_{TI_s}(n, \perp \perp \perp)$  defines a primitive recursive predicate.

2. By Theorem 14.4, we have  $TI_{s+1} \vdash Con(SLP_s)$ . By Theorem 12.1.1,  $SLP_{s+1} \vdash int(Con(SLP_s))$  and by Lemma 12.3,  $SLP_{s+1} \vdash Con(SLP_s)$ .  $\dashv$

**THEOREM 14.6.** 1.  $SLP_2 \vdash Con(I_n)$  for  $n = 0, 1, 2, \dots$ .

2.  $SLP_3 \vdash Con(I)$ .

3.  $SLP \vdash Con(I)$ .

**PROOF.** 1. By Theorem 13.3, we have  $TI_2 \vdash Con(I_n)$ . By Theorem 12.1.1,  $SLP_2 \vdash int(Con(I_n))$  and by Lemma 12.3,  $SLP_2 \vdash Con(I_n)$ .

2. By Theorem 13.4.1,  $TI_3 \vdash Con(I)$ . By Theorem 12.1.1,  $SLP_3 \vdash int(Con(I))$  and  $SLP_3 \vdash Con(I)$  by Lemma 12.3.

Part 3 follows from part 2.  $\dashv$

The theory  $SLP$  is clearly weaker than the fully impredicative theory  $T$  but it has the same strength as the quite impredicative theory  $TI$  and is stronger than the more predicative theory  $I$ . In particular,  $SLP$  is stronger than the second order arithmetic.

**§15. Discussion.** In this paper we constructed a Beth model  $\mathcal{B}_s$  for the language of intuitionistic functionals of types  $1, 2, \dots, s$ . We combined the intuitionistic principles that hold in the models  $\mathcal{B}_s (s \geq 1)$  into a relatively strong intuitionistic theory  $SLP$ . The theory  $SLP$  contains axioms for lawless functionals of high types, some versions of choice axioms and the theory of the "creating subject".

We did not manage to construct one Beth model for the entire theory  $SLP$ ; combining all the models  $\mathcal{B}_s$  in one does not work because their cardinalities increase indefinitely with the increase of  $s$ . However, the models  $\mathcal{B}_s (s \geq 1)$  are sufficient, since any proof in  $SLP$  is finite and contains only a finite number of types.

We compared  $SLP$  with a classical typed set theory  $TI$  that has a restricted but impredicative comprehension axiom. By formalising the Beth model  $\mathcal{B}_s$  in the language  $TI$ , we showed the equiconsistency of each fragment  $TI_s$  with the corresponding fragment  $SLP_s$  and also the equiconsistency of  $TI$  and  $SLP$ . Thus,  $SLP$  is stronger than any fragment  $TI_s$  and in particular, it is stronger than the second order arithmetic.

We interpreted the classical theory  $TI$  in our intuitionistic theory  $SLP$ . Many parts of classical mathematics can be developed in  $TI$  and therefore justified from the intuitionistic point of view, due to this interpretation.

## REFERENCES

- [1] S. BERNINI, *A very strong intuitionistic theory*, **Studia Logica**, vol. 35 (1976), no. 4, pp. 377–385.
- [2] ———, *A note on my paper “A very strong intuitionistic theory”*, **Studia Logica**, vol. 37 (1978), no. 4, pp. 349–350.
- [3] A. G. DRAGALIN, **Mathematical intuitionism. Introduction to proof theory**, American Mathematical Society, USA, 1987.
- [4] H. FRIEDMAN, *Set theoretic foundations for constructive analysis*, **Annals of Mathematics (2)**, vol. 105 (1977), no. 2, pp. 1–28.
- [5] F. KACHAPOVA, *A generalization of Beth model to functionals of high types*, **Proceedings of the 12th Asian Logic Conference**, (2013), pp. 185–209.
- [6] F. KASHAPOVA, *Intuitionistic theory of functionals of higher type*, **Mathematical Notes**, vol. 45 (1989), no. 3, pp. 66–79.
- [7] A. N. KOLMOGOROV and A. G. DRAGALIN, **Introduction to mathematical logic**, Moscow University, Moscow, 1982.
- [8] V. A. LYUBETSKI, *Transfer theorems and intuitionistic set theory*, **Dokl. Akad. Nauk**, vol. 357 (1997), no. 2, pp. 168–171.
- [9] R. MCNAUGHTON, *Some formal relative consistency proofs*, **The Journal of Symbolic Logic**, vol. 18 (1953), no. 2, pp. 136–144.
- [10] E. MENDELSON, **Introduction to mathematical logic**, Chapman and Hall/CRC, 2009.
- [11] J. MYHILL, *Formal systems of intuitionistic analysis*, **Logic, Methodology and Philos. Sci. III**, (1968), pp. 161–178.
- [12] D. VAN DALEN, *An interpretation of intuitionistic analysis*, **Annals of Mathematical Logic**, vol. 13 (1978), pp. 1–43.
- [13] N. WENDEL, *The inconsistency of Bernini’s very strong intuitionistic theory*, **Studia Logica**, vol. 37 (1978), no. 4, pp. 341–347.

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